2.1 Probabilities at a Time and in Time

Within quantum mechanics, a complete set of commuting observables can be found hich describe the attributes of a system at a given time. Ho ever, difficulties arise for attributes of a system that extend over time, such as the time of an atomic decay, the time of arrival, etc. As e discuss belo , simple extensions of ordinary notions of probabilities *at a certain time* to probabilities *in time* give rise to distributions hich can no longer be interpreted as probabilities. The reason for this can be understood in simple terms. Consider for example, the event of a particle entering a box. What is the time of the event? Classically, there is no distinction bet een attributes at one time or in time. One can, for example, measure the position and momentum of the particle at any time ith negligible disturbance and use this information to deduce the time of entering the box. Quantum mechanically, ho ever, there are t o separate questions. We can either ask at a certain time t_0, "has the particle already entered the box?", or e can ask " hen did the particle enter the box?" To ans er the first question e simply measure at time t_0 if the particle is in the box. Although quantum mechanics does provides us ith a prediction for the probability $P(t_0)$ for this event, this probability does not describe a probability in time. The measurement at time t_0 disturbs the evolution of the system in the future and hence the probability distribution at time $t > t_0$ ill no longer be given by $P(t)$. In fact by the same consideration, e see that the t o questions above, or the corresponding measurements, are not compatible ith each other, in the sense that one measurement disturbs the other and visa-versa. One ould think that the questions " hen did an event occur?" and "has the event occurred?" can be ans ered simultaneously, ho ever, as e shall see, they are in fact complementary. One cannot necessarily ans er both questions simultaneously. In the next section e ill formulated these difficulties in a more precise ay.

We ill then examine t o specific cases of measuring the time of an event. One is the arrival of a particle to a certain location, and another is a recent proposal of Rovelli [28] to measure the time that a measurement occurred. We argue that his scheme only ans ers the first question: "has the measurement occurred already at a certain time?", but does not ans er the more difficult question " hen did the measurement occur?" In other ords, it does not provide a proper probability distribution for the time of an event. We also discuss the relationship bet een Rovelli's measurement scheme, and the use of the probability current for measurements of time-of-arrival. In Section 2.4 e discuss a

For the case of time-of-arrival, $\mathbf{\Pi}_a$ ill be the projector onto a region of the x-axis (in this case, the index i is continuous).

If initially the system is in the state ψ then in the Heisenberg representation the probability that the event has happened at any time t is given by

$$P_a(t) = \langle\psi|\mathbf{\Pi}_a(t)|\psi\rangle \ . \tag{2.6}$$

One can also compute the "current operator"

$$\mathbf{J}_a = \frac{d\mathbf{\Pi}_a(t)}{dt} \tag{2.7}$$

 hich gives the rate of change of the probability distribution $P_a(t)$. It is tempting to argue that the probability distribution

$$p_a(t) = \langle\psi|\mathbf{J}_a(t)|\psi\rangle \tag{2.8}$$

gives the probability that the event a happens bet een t and dt, since classically the probability that an event happened some time before time t is just the integral bet een some initial time t_o and t of the probability that the event happens at that time.

Ho ever, the probability distribution obtained from $\mathbf{J}_a$ cannot be thought of as a probability distribution *in time*. $p_a(t)$ is not the probability that the event happened at time t. To see that $p_a(t)$ is not a probability distribution in time, let us compare its properties to the properties (1-4) of the conventional quantum [1] probability distribution obtained from the projectors $\mathbf{\Pi}_i$.

Property 1 *The probability of finding that the system is in the state ϕ_i at time t is independent of the probability of finding that the system is in the state ϕ_j (at the same time t).*

[1] properties 2-4 are also true of classical probability distributions

i.e..

$$[\mathbf{\Pi}_i(t), \mathbf{\Pi}_j(t)] = 0. \tag{2.9}$$

If e interpret the probabilities $P_a(t)$ as probabilities in time, then our conventional notions of hat these probabilities mean, break do n. In general,

$$[\mathbf{\Pi}_a(t), \mathbf{\Pi}_a(t')] \neq 0. \tag{2.10}$$

Measurements made at earlier times influence measurements made at later times. The possible results of an observable at time t ill depend on hether there ere any previous measurements of that observable. In classical mechanics, one can make the interaction of the measuring device ith the system arbitrarily eak, and therefore, not disturb the evolution of the system in time, but this is not true in quantum mechanics. A measurement of position at t_1 for example, ill disturb the momentum of the particle in such a ay that future measurements of position at t_2 ill give very different results from the case here no measurement as performed at t_1. Since $\mathbf{\Pi}_a(t)$ does not commute ith itself at different times, there is no reason to believe that $\mathbf{J}_a$ ill commute ith itself at different times either. It is essentially this difference bet een conventional probabilities and those obtained from $\mathbf{\Pi}_a$ hich prevents us from determining hen an event occurred.

In addition, $p_a(t)$ and $P_a(t)$ do not have the follo ing other properties of quantum distributions:

Property 2 *If $i \neq j$ then the projection operators project onto orthogonal states.*

i.e..

$$\mathbf{\Pi}_i(t)\mathbf{\Pi}_j(t) = 0 \quad i \neq j \ . \tag{2.11}$$

For example, if a particle is found at position x, then it could not have been any here else at the same time. On the other hand, a particle may be at the same position at many different times. There is no reason hy the event a can not happen at many different to only act on states for hich $\mathbf{P}_a$ is increasing ith time, but the restricted domain of definition of $\mathbf{J}_a$ may mean that it ill no longer be self-adjoint. Furthermore, hether $\mathbf{J}_a$ is positive or negative ill not only depend on the state, but also on the Hamiltonian. For certain Hamiltonians, one may find that there are no states for hich $p_a(t)$ does not take on negative values.

Another interesting aspect of $\mathbf{J}_a$ and $\mathbf{\Pi}_a$ is that in general

$$[\mathbf{J}_a(t), \mathbf{\Pi}_a(t)] \neq 0. \tag{2.15}$$

The operator hich measures that the event happened and the operator $\mathbf{J}_a$ do not commute. If one believes that $\mathbf{J}_a$ can be used to ans er the question " hen did the event happen?" then one finds that " hen did it happen?" and "has it already happened?" seem to be complimentary (in Bohr's sense) in that they interfere ith each other. Naively, it ould seem that determining " hen did a occur?" ould also ans er the question "has a occurred?". Ho ever the inaccuracy of the determination of " hen did a occur?" seems to place limits on our ability to ans er "has a occurred?".

2.3 Time of a Measurement or Arrival

We no examine t o specific examples of the determination of hen an event occurred.

here **P** is the conjugate momentum to the pointer **Q** of the measuring device, and $g(t)$ is a function hich is zero every here, except during a small interval of time. After the

In the case of time-of-arrival, one ishes to measure the time a particle arrives to a certain location (say $x = 0$). Often, the probability current is used to determine the arrival time[13]. One imagines that a particle is localized in the region $x < 0$ and traveling to ards the origin. The projector

$$\mathbf{\Pi}_+ = \int_0^\infty dx|x\rangle\langle x| \tag{2.21}$$

is an operator hich is equal to one hen $x > 0$ and zero other ise. The probability of detecting the particle in the positive x-axis is given by

$$P_+(t) = \langle\psi|\mathbf{\Pi}_+(t)|\psi\rangle. \tag{2.22}$$

In the Schrödinger representation, this expression is just $P_+(t) = \int_0^\infty |\psi(x,t)|^2 dx$. It is then claimed that the current $\mathbf{J}_+$, given by

$$\frac{\partial \mathbf{J}_+}{\partial x} = \frac{d\mathbf{\Pi}_+(t)}{dt} \tag{2.23}$$

ill give the probability that the particle arrives bet een t and $t + dt$.

It is clear that both the operators $\mathbf{M}(t)$ and $\mathbf{\Pi}_+(t)$ are specific example of the operator $\mathbf{\Pi}_a$ discussed in Section 2.2. $\mathbf{M}(t)$ gives the probability at time t that a measurement has occurred. $\mathbf{\Pi}_+(t)$ gives the probability that the particle is found at $x > 0$ at time t. The t o operators $\mathbf{m}(t)$ and $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ are examples of $\mathbf{J}_a(t)$. $\mathbf{m}(t)$ gives the change in the probability that the measurement happened at time t, hile $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ gives the change in the probability that the particle is found at $x > 0$. Ho ever, one cannot interpret these operators as giving the probability that the measurement occurred (or the particle arrived). None of these operators allo one to measure the precise time at hich the event occurred. They do not posses all the Properties 1 therefore depend on t. For $t - t' \ll d\mathbf{H}$ e have for any operator $\mathbf{A}(t) \simeq \mathbf{A}(t') + i(t - t')[\mathbf{H}, \mathbf{A}(t')]$, and so

$$[\mathbf{A}(t), \mathbf{A}(t')] \simeq i(t - t')[\mathbf{H}, \mathbf{A}(t')], \mathbf{A}(t')] \tag{2.24}$$

For arbitrary Hamiltonians, it is obvious that none of the operators above ill commute ith themselves at different times. Even for a free particle, one can explicitly calculate that neither $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ nor $\mathbf{\Pi}_+(t)$ commute ith themselves at different times, (the calculation is neither difficult, nor particularly illuminating).

For some very specific states, and physical situations, Properties 2-4 may be obeyed, but this is certainly not true in general. For the case of time-of-arrival, even for a free Hamiltonian and ave packets hich only contain modes of positive frequency, the current can be negative [29] (a violation of Property 4 - that probabilities must be positive definite). In fact, since the current is simply the time derivative of a projection operator, there is no reason to expect it to al ays be positive. For free particles hich can arrive from the left and right, the current can be zero [2] and hence the probability distribution ill be unnormalizable (Property 3). Also, in general, there is no reason hy a particle can't be at the same position at many different times (a violation of Property 2). In the case of particles hich move in a potential, one may find that there are no states for hich Properties 3-4 are obeyed. For example, if there is an infinite potential barrier around the origin, the particle ill never arrive, and the current ill not be normalizable, and if there is a harmonic oscillator potential, the particle ill cross the origin many times from both the left and the right violating Properties 2 and 4. For the case of determining hen a measurement occurred, Rovelli restricts the class of measurements he considers to be those for hich $\mathbf{m}(t)$ obeys Properties 2-4. As a result, $\mathbf{m}(t)$ cannot be used for arbitrary measurements. As ith the time-of-arrival, there are clearly many Hamiltonians

[2] See Appendix A where we see that a coherent antisymmetric superposition of left and right moving waves has zero current.

for hich Property 2-4 ill be violated. Nor can $\mathbf{m}(t)$ be used for Hamiltonians for hich its restricted domain of definition ill mean that it is no longer self-adjoint.

Although operators such as $\mathbf{m}$ and and $\mathbf{J}_+$ do not commute ith themselves at different times, it is possible to construct an operator hich is time-translation invariant, and ould give the time of an event in the classical limit. This ill be discussed in Chapter 4 here e shall see that such an operator cannot be self-adjoint if the Hamiltonian is bounded from above or belo .

2.4 Continual Event Monitoring

Instead of considering operators, a more physical meaningful method of measuring the occurrence of an event is to consider continuous measurement processes. For example, the operator $\mathbf{\Pi}_a(t)$ can be measured continuously or at small time intervals. When one considers such a physical measurement procedure one can see that the time at hich an event occurs is not ell defined in quantum mechanics. The probability of finding that the system enters one of the states ϕ_i at time t_a is given by the probability that it isn't in any of the states ϕ_i before t_a, times the probability that it is in one of the states ϕ_i at t_a.

To see ho such a scheme might ork, let us see ho one ould measure the time of an occurrence of the event corresponding to $\mathbf{\Pi}_a$. A measurement of the operator $\mathbf{\Pi}_a(t)$ ill tell us hether the event has occurred at time t. We can then measure $\mathbf{\Pi}_a(t)$ at times $t_k = k\Delta$ for integral k in order to determine hen the measurement occurred. Δ represents the frequency ith hich e monitor the system, and is therefore the inaccuracy of the measurement in time (it is the coarseness of the measurement in some sense).

We ill no ork in the Schrödinger representation, simply because it is the most natural arena to talk about successive measurements on a system. At time t_1, the probability that an event has occurred is given by

$$P(\uparrow, t_1) = \langle\psi_0(0)|\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}|\psi_0(0)\rangle \tag{2.25}$$

and the probability that it hasn't is

$$P(\downarrow, t_1) = \langle\psi_0(0)|\mathbf{U_\Delta}^\dagger (\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0(0)\rangle \tag{2.26}$$

here $\uparrow$ corresponds to detecting that the event has occurred, $\downarrow$ corresponds to detecting that an event has not yet occurred, $\psi_0(0)$ is the initial state of the system and $\mathbf{U_\Delta}$ is the evolution operator e^{-i}

and the probability that an event hasn't occurred is

$$P(\downarrow, t_k) = \langle\psi_0|B_k|\psi_0\rangle \tag{2.32}$$

ith

$$B_k = \mathbf{U_\Delta}^\dagger(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1}-\mathbf{\Pi}_a)...\mathbf{U_\Delta}^\dagger(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta}...(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1}-\mathbf{\Pi}_a)\mathbf{U_\Delta} \tag{2.33}$$

By allo ing the unitary operators to act on the projection operators e can rite the A_k or B_k in the Heisenberg representation. For example

$$A_k = (\mathbf{1}-\mathbf{\Pi}_a)(t_1)...(\mathbf{1}-\mathbf{\Pi}_a)(t_{k-1})\mathbf{\Pi}_a(t_k)(\mathbf{1}-\mathbf{\Pi}_a)(t_{k-1})...(\mathbf{1}-\mathbf{\Pi}_a)(t_1) \tag{2.34}$$

HO ever, hil B_k the operators $\mathbf{\Pi}_a(t)$ can b e operatoo pd by p **T**ary eme- lolu e j of bH here $\int g(t)dt = \pi$ ($g(t)$ is sharply peaked, ith idth T), and the primed Pauli matrix acts on the measuring device, hile the unprimed Pauli matrix acts on the system. After a time T, the spin of the measuring device ill be correlated ith the system. Since this measurement is rather crude, (the initial state of the device is the same as one of the measurement states), the operator $\mathbf{M}$ at $t = 0$ is not zero. Let us simplify the problem further, by assuming that $a = 0$ and $b = 1$. In this case, the only relevant matrix element of $\mathbf{1} - \mathbf{M}$ is $|\downarrow\rangle \otimes |\uparrow'\rangle\langle\uparrow'| \otimes \langle\downarrow| = |\psi_o\rangle\langle\psi_o|$. We then find the probability that the measuring apparatus has not responded at time t_k is

$$\begin{aligned} P(\downarrow, t_k) &= |\langle\psi_o|U_\Delta|\psi_o\rangle|^{2k} \\ &\simeq |\langle\psi_o|1 - i\Delta\mathbf{H} - \Delta^2\mathbf{H}^2|\psi_o\rangle|^{2k} \\ &\simeq 1 - \Delta^{2k}(\langle\mathbf{H}^2\rangle - \langle\mathbf{H}\rangle^2)^k \end{aligned} \tag{2.37}$$

If e fix a value of $\tau = t_k$ and then make Δ go to zero, e find

$$\begin{aligned} P(\downarrow, \tau) &\simeq e^{-(\Delta dE)^2\tau/\Delta} \\ &\simeq 1, \end{aligned} \tag{2.38}$$

hich implies that the measuring apparatus becomes frozen and never records a measurement. In order not to freeze the apparatus, e need $\Delta > 1/dE$ here dE is the uncertainty in energy of the measuring device O (initially the spacing bet een energy levels in this case). There is al ays an inherent inaccuracy hen measuring the time that the event (of the measurement) occurred. This inaccuracy is similar to the one hich e ill find in Chapter 3. Note that as discussed in the Introduction, this inaccuracy is not related to the so-called "Heisenberg energy-time uncertainty relationship" as it applies to every single measurement and not to the idth of measurements carried out on an ensemble.

One can of course use the set of operators A_k to compute a probability distribution in time, or experimentally determine a probability distribution for the time of an event. Ho ever, as e have just seen, this probability distribution is not a function of the system alone, but rather, it is related to the system and the measuring device (or set of operators) For example, the probability distribution ill depend on Δ, and if Δ is too small, e ill find that the event never occurs. The distribution $P(\uparrow, t_k)$ does allo as to predict the probabilities of future measurements using a particular measuring device, but the results are not attributes of the system.