

QUANTUM TIME

By

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Abstract

In quantum mechanics, time plays a role unlike any other observable. We find that measuring whether an event happened, and measuring when an event happened are fundamentally different – the two measurements do not correspond to compatible observables and interfere with each other. We also propose a basic limitation on measurements of the arrival time of a free particle given by $1/\bar{E}_k$ here \bar{E}_k is the particle's kinetic energy. The temporal order of events is also an ambiguous concept in quantum mechanics. It is not always possible to determine whether one event lies in the future or past of another event. One cannot measure whether one particle arrives to a particular location before or after another particle if they arrive within a time of $1/\bar{E}$ of each other, here \bar{E} is the total kinetic energy of the two particles. These new inaccuracy limitations are dynamical in nature, and fundamentally different from the Heisenberg uncertainty relations. They refer to individual measurements of a single quantity. It is hoped that by understanding the role of time in quantum mechanics, we may gain new insight into the role of time in a quantum theory of gravity.

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Dedication

In loving memory of my father, Peter Oppenheim (1942-1998) - my first physics teacher, who encouraged my curiosity, patiently answered my questions, and patiently asked his opinion. He would have loved to flip through this thing, and I had always imagined giving him a copy.

Acknowledgements

I would like to thank my Ph.D. advisor, Bill Unruh, not only for teaching me a great deal of physics, but also for teaching me how to attack problems and discover interesting questions. For allowing me to wander off in other directions and subtle guidance.

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Much of this work is contained in [1] - [7], and was done in collaboration with Yakir Aharonov, Benni Reznik, Sandu Popescu, and Bill Unruh. Thanks to my collaborators for their help, for many interesting discussions, insights and paradoxes.

Throughout my work, I have had the benefit of discussions with a number of researchers. Carlo Rovelli, Gonzalo Muga, Rick Leavens, and Bob Wald have been particularly helpful.

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Special thanks to the amazing Victoria Scott for drawing the cartoons for this thesis, as well as for everything else.

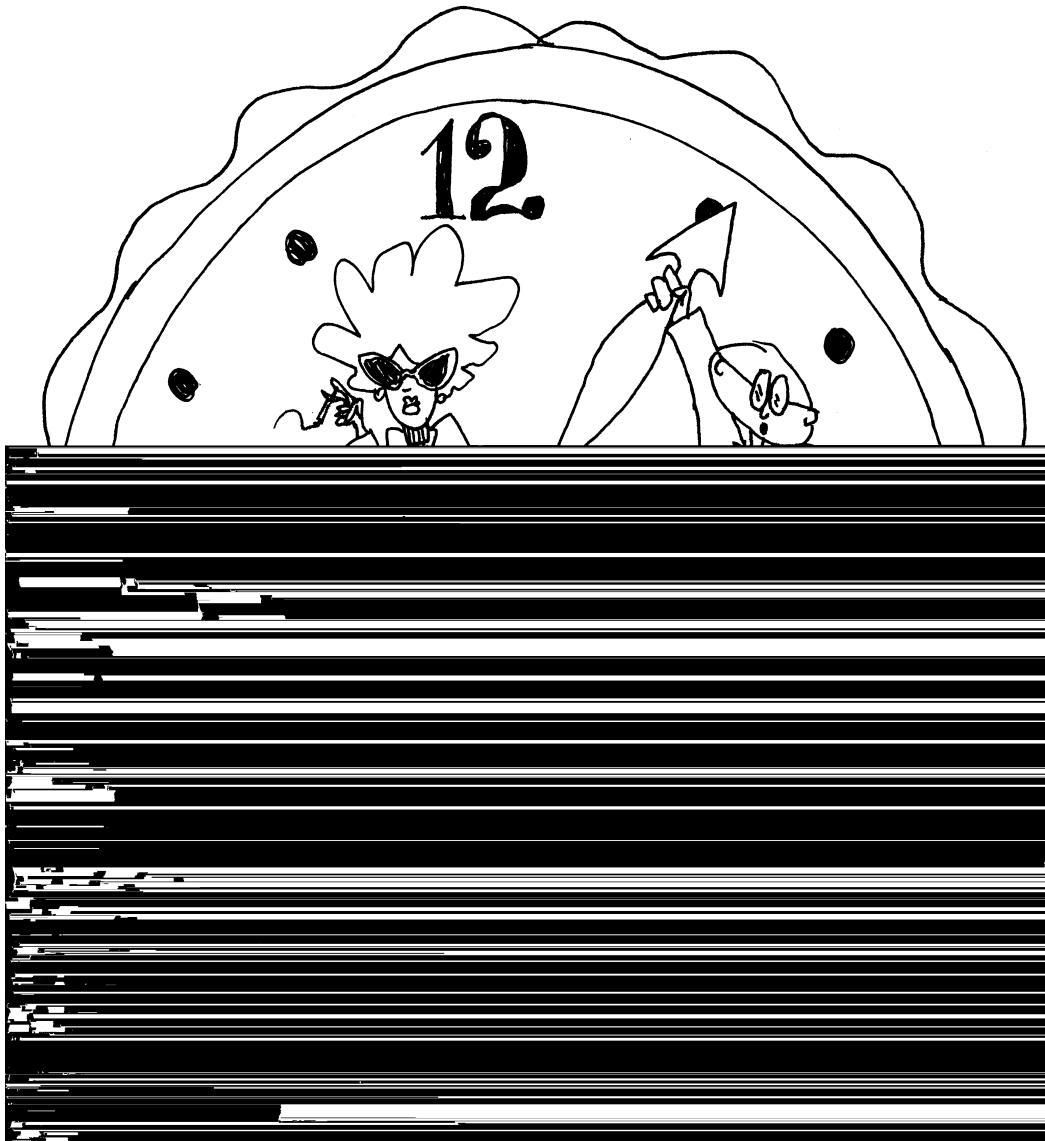
Thanks to all my friends and house mates for putting up with me.

To my parents, my brother and the grannies, I thank them for their encouragement their support and their interest, for giving me red boxes to play with, reading to me, and killing the television.

And finally, no thanks to APEC, the Big Cheese, and the Mounties.

Chapter 1

Introduction



1.1 Dual Measurements

One of the first lessons of quantum mechanics is that a property of a system does not correspond to an element of reality until it is measured. It makes no sense to talk about the position of a particle or the momentum of the particle, in and of itself. It is only when we measure a physical quantity that we can actually say that a system possesses it. The particle does not have a position until its position is actually measured.

Ordinarily in quantum mechanics, one is interested in measuring properties of a system at a particular time t . One might want to know a particle's position, momentum, or spin, and the measurement of this quantity occurs at a certain time. For experiments at a fixed time, quantum mechanics provides us with a useful formalism to describe reality. Observables are represented by self-adjoint operators, and in the Heisenberg representation they evolve in time. The possible results of any measurement at any instant of time t can be found by applying these operators to the wave function of the system at a particular time.

the time becomes the observable one is trying to measure.

Classically, the time of an event can be made into an observable just like any other and this time can be measured in a variety of ways, all of which give the same result. One can simply invert the equations of motion of the system to find the time that an event occurs¹, and then measure the values of the canonical variables (generalized coordinates and conjugate momenta). Since classically there is no uncertainty relation preventing the measurement of all the coordinates and conjugate momenta simultaneously, there is no limitation for finding the event's time. One could also continually monitor the system to determine the precise time when the event occurred. Since one can make the interaction between the system and the measuring apparatus as small as one likes, this measurement need not disturb the evolution of the system. Finally, one can also couple a clock to the system in such a way that the clock stops when the event occurs. All these methods yield the same results, and work to any desired accuracy.

Dual measurements, are quite common in modern laboratory experiments. In particle physics one often wants to know the time that certain collisions or decays occurred. However, surprisingly, dual measurements are not easily dealt with using the conventional tools of quantum mechanics.

Pauli [8] was the first to demonstrate that there was no operator associated with time. A time operator must be conjugate to the Hamiltonian, and he proved that this is impossible if the Hamiltonian for the system is bounded from above or below. The reason for this is that an operator which is conjugate to the Hamiltonian acts as a shift operator for energy, and one could use it to shift the energy below any lower bound (or above any upper bound).

Since then, there have been numerous attempts to circumvent his proof by considering

¹In some systems (especially in the context of general relativity), it is only possible to find the time locally. A global time variable may not exist.

the time-of-arrival.

The interest in a quantum mechanical time operator stems in part from the troubling

It is space-time which is the element of reality in general relativity.

These coordinates are, of course, subject to coordinate transformations, and in particular, the theory is invariant under reparametrization of the time coordinate. One consequence of this, is that if one tries to canonically quantize Einstein's theory of gravity in a closed system, one finds that the wave-function must satisfy the Wheeler-DeWitt equation

$$\mathcal{H}\Psi(g_{ab}, \pi_{ab}) = 0 \tag{1.1}$$

here the wave function depends on the 3-metric and conjugate momenta and \mathcal{H} is

exist many ambiguities in the role of time in quantum mechanics. Our hope is that a better understanding of time in the arena of quantum mechanics will benefit and inform research in the field of quantum gravity. At the end of this thesis, we will discuss some of the connections between the problem of time in quantum gravity and our research.

1.2 Differences Between Measurements of Space and Measurements of Time

Ever since Einstein's theory of special relativity, we have been encouraged to think of time and space on an equal footing. However, even classically, time and space are quite different as our common experience tells us. Objects move constantly forward in time in a manner very different to the way they move through space. Although we will discuss in more detail the differences between quantum measurements of ordinary observables and measurements of time in Chapter 2, it may be instructive to roughly outline the differences between measurements of a particle's position at a fixed time, and the time a particle is found at a particular location.

In standard quantum mechanics, the probability that a particle is found at a given location X at time t is given by

$$P_t(X) = |\psi(X, t)|^2 . \quad (1.2)$$

If we know $\psi(x, 0)$ for all x then the system is completely described and we can easily compute this probability distribution at an instant of time. If we know the Hamiltonian of the system, then using the Schrödinger equation we can also compute $\psi(x, t)$ at any time t . This probability distribution corresponds to results of a measurement of position at a particular time. Quantum mechanics gives a well defined answer to the question, “where is the particle at time t ?”

However, it is also perfectly natural to ask “at what time is the particle at a certain location.” Here, quantum mechanics does not seem to provide an unambiguous answer.

At first sight it seems that the probability distribution $P_x(T)$ to find the particle at a certain time at the location x is simply $|\psi(x, T)|^2$. However, $|\psi(x, T)|^2$, does not represent a probability *in time*, since it is not normalized with respect to T .

One might be tempted therefore, to consider the quantity

$$P_x(T) = \frac{|\psi(x, T)|^2}{\int |\psi(x, t')|^2 dt'} \quad (1.3)$$

This normalization depends on the particular state being measured, and can only be done if one knows the state $\psi(x, t)$ at all times t (infinitely far in the past and future). There are also states for which the particle is never found at the position x , in which case the expression above is undefined. Notwithstanding this, one might argue that this quantity gives one a relative probability that the particle is found at the location x at time T (if the measurement is made at that time T), as opposed to another time T' (if the measurement is made at time T').

However, the expression above certainly does not yield the probability *in time* to detect the particle. One reason for this failure is that a particle may be detected at a location X at many different times t (e.g. I can be found in my office at many different times in the day). On the other hand, if at time t a particle is detected at location X , then we can say with certainty that at the same time t , the particle was not at any other location X' (e.g. at nine a.m. I am in bed, and therefore, I cannot also be in my office). Equation (1.3) does not give a proper probability distribution as the various outcomes are not disjoint. $P_x(T)$ is not a probability distribution in time in the sense usually reserved for probability distributions in quantum mechanics. $P_x(T)$ is very different from $P_t(X)$ and has different properties (as we will see in the next chapter).

This leads us to consider the time of first arrival of a particle t log t e rs . M (. U fh

the particle as not there at any previous time. In other words, one must continuously monitor the location x_A in order to find out when the particle arrives. However, this continuous measurement procedure has its own difficulty, and also emphasizes the problem with the previous probability distribution. Namely, that the probability to find a particle at $t = T$ is generally *not* independent of the probability to find the particle at some other time $t = T'$. i.e.. if Π_{x_A} is the projector onto the position x_A , then in the Heisenberg representation ³

$$[\Pi_{x_A}(t), \Pi_{x_A}(t')] \neq 0. \quad (1.4)$$

Measurements made at different times disturb each other. We will see in Section 2.2 that this is one of the properties of ordinary measurements which measurements in time violate. Measurements made at different times do not commute. Therefore the probability distribution obtained from this measurement procedure, although well defined, does not give a probability distribution *in time*.

Von Neumann measurements ⁴ happen *at a certain time*. One measures the particle's position at time t

laar coheren s i a l s tmes ty

of detecting the particle at some other time t' .

1.3 Inaccuracies and Uncertainties

The measurement of an observable corresponding to a self-adjoint operator can be as accurate as one wishes. This is true despite any uncertainty relations which govern various sets of observables. The position, or momentum of a particle (but not both) can be measured to any desired precision. Consider two observables \mathbf{A} and \mathbf{B} which do not evolve in time, and whose commutator is i (in units where $\hbar = 1$). Imagine that we have an ensemble of identical systems prepared in some initial state. On half the ensemble, we can measure \mathbf{A} , and on the other half, we can measure \mathbf{B} . Each individual measurement can be as accurate as we wish. An extraordinary experimentalist can reduce the inaccuracies in the measurement to almost zero, and can get a particular value for each measurement. The experimentalist may have a dial on her device which will point to the value of A after the measurement. She will have to make sure that initially the pointer on her dial points almost exactly to zero, and then after each run of her experiment, she measures the position of the dial very accurately to determine the value of A .

If we then plot all of the measurements of \mathbf{A} and all of the measurements of \mathbf{B} , we will find a distribution of measurements which have a natural width of ΔA and ΔB respectively. One then finds that no matter what initial state we choose, $\Delta A \Delta B > 1$. There is an *uncertainty* relation between the distributions of A and B , but there are no theoretical limitations on the accuracy of each individual measurement of \mathbf{A} or \mathbf{B} .

The experimentalist does not have to make her measurements totally precise. She could, for example, start off the experiment with her dial in a state where the initial position of the needle is *uncertain*. An uncertainty in the initial pointer position will result in her measurement being *inaccurate*. When she measures the final position of her

of the probability current to measure the time at which a particle arrives to a certain location. The discussion suggests that the difference between time and other observables is not merely formal.

The central result of the thesis is contained in Chapter 3 where we discuss the problem of the time-of-arrival of a particle to a particular location. It is argued that the time-of-arrival cannot be precisely defined and measured in quantum mechanics. By constructing explicit toy models of a measurement involving physical clocks, we show that the time-of-arrival for a free particle cannot be measured more accurately than $\delta t_A \sim 1/\bar{E}_k$, where \bar{E}_k is the initial kinetic energy of the particle. With a better accuracy, particles reflect off the measuring device and the resulting probability distribution becomes distorted. This is a new result.

relationship between these modified operators, and the direct measurements discussed in Chapters 2 and 3, and argue that a measurement of the time-of-arrival operator does not correspond to these continuous measurements. Unlike the classical case, in quantum mechanics the result of a measurement of the time-of-arrival operator may have nothing to do with the time-of-arrival to $x = x_A$.

There has been renewed interest in time-of-arrival operators following the suggestion by Grot, Rovelli, and Tate, that one can modify the low momentum behavior of the operator slightly in such a way as to make it self-adjoint [9]. We show that such a modification results in the difficulty that the eigenstates are drastically altered. In an eigenstate of the modified time-of-arrival operator, the particle, at the predicted time-of-arrival, is found far away from the point of arrival with probability $1/2$.

The bound of $1/\bar{E}_k$ on the accuracy of time-of-arrival measurements is based on calculations done using numerous measurement models corresponding to specific Hamiltonians, as well as more general considerations. However, because the limitation is based on dynamical considerations and not kinematic ones, a formal proof of the limitation may not exist. For example, a proof of the Heisenberg uncertainty relation relies only on the properties of specific operators, while our inaccuracy relation is a statement not about operators, but about measurements (and therefore, involves the dynamical considerations of the actual measurement). Perhaps by making certain restrictive assumptions about the Hamiltonian one might be able to construct a formal proof. Such a proof would have to take into account the measurement model which will be discussed in Section 3.3.3 in which we show that if one has prior information about the wavefunction, and if the wavefunction is almost an eigenstate of energy (i.e. its time of arrival is completely uncertain), then one can measure the time of arrival to an accuracy better than $1/\bar{E}_k$. One therefore expects that a formal proof will not only have to involve making assumptions about the interaction Hamiltonian, but also the initial state of the wave function. The existence of

a formal proof for our inaccuracy limitation remains an interesting open question.

While we know of no formal proof for the inaccuracy limitation for time-of-arrival, one can make more general statements about measurements of "traversal time". In Chapter 5 we consider the problem of a free particle which traverses a distance L and argue that a violation of the above limitation for the traversal-time implies a violation of the Heisenberg uncertainty relation for x and p . This result does not depend on the details of the model being used in the measuring process. Measurements of traversal-time are dual to measurements of traversal distance, and it can be shown that one can measure the distance a particle travels to any desired precision. This chapter also contains a further discussion on the difference between what we call "inaccuracy" limitations, which constrain the precision with which individual measurements are performed, and "uncertainties" which are kinematic quantities which relate to the spread in measurements on ensembles.

Chapter 6 contains what may be our most interesting result. In it, we examine whether one can determine the temporal ordering of events. We find that one cannot measure whether one event occurred in the future or past of another event to arbitrary accuracy. The minimum inaccuracy for measuring whether a particle arrives to a given location before or after another particle is given by $1/\bar{E}$ where \bar{E} is the total kinetic energy of the two particles. We discuss the relationship between this type of measurement, and coincident counters, as well as Heisenberg's microscope. We show that in general one cannot prepare a two particle state where the two particles always arrive within a time of $1/\bar{E}$ of each other. This has interesting consequences for determining the metric properties of a space-time.

In this thesis we will work in units where $\hbar = c = 1$

Chapter 2

When does an Event Occur

When did Schrödinger's cat die?



2.1 Probabilities at a Time and in Time

Within quantum mechanics, a complete set of commuting observables can be found which describe the attributes of a system at a given time. However, difficulties arise for attributes of a system that extend over time, such as the time of an atomic decay, the time of arrival, etc. As we discuss below, simple extensions of ordinary notions of probabilities *at a certain time* to probabilities *in time* give rise to distributions which can no longer be interpreted as probabilities. The reason for this can be understood in simple terms. Consider for example, the event of a particle entering a box. What is the time of the event? Classically, there is no distinction between attributes at one time or in time. One can, for example, measure the position and momentum of the particle at any time with negligible disturbance and use this information to deduce the time of entering the box. Quantum mechanically, however, there are two separate questions. We can either ask at a certain time t_0 , “has the particle already entered the box?”, or we can ask “when did the particle enter the box?”

We will then examine two specific cases of measuring the time of an event. One is the arrival of a particle to a certain location, and another is a recent proposal of Rovelli [28] to measure the time that a measurement occurred. We argue that his scheme only answers the first question: “has the measurement occurred already at a certain time?”, but does not answer the more difficult question “when did the measurement occur?” In other words, it does not provide a proper probability distribution for the time of an event. We also discuss the relationship between Rovelli’s measurement scheme, and the use of the probability current for measurements of time-of-arrival. In Section 2.4 we discuss a model and set of operators which can be used to determine a probability distribution for the time of an event. The model is based upon a continuous process akin to a rapid series of measurements. We find that in the limit of high accuracy the system is severely disturbed and the measurement does not work, an effect which is analogous to the Zeno paradox.

2.2 Did it Occur vs. When Did it Occur

In conventional quantum mechanics, for each observable, we can assign a set of projection operators Π_i onto a set of eigenstates ϕ_i of some operator. At each time t there exists a Hilbert space and inner product which enable one to calculate the probability $P_i(t)$ that the system is in one of the states ϕ_i . In certain cases, one can find a subset a of the set i such that the projection operator

$$\Pi_a = \sum_{i \in a} \Pi_i$$

For the case of time-of-arrival, Π_a will be the projector onto a region of the x-axis (in this case, the index i is continuous).

If initially the system is in the state ψ then in the Heisenberg representation the probability that the event has happened at any time t is given by

$$P_a(t) = \langle \psi | \Pi_a(t) | \psi \rangle . \quad (2.6)$$

One can also compute the “current operator”

$$\mathbf{J}_a = \frac{d\Pi_a(t)}{dt} \quad (2.7)$$

which gives the rate of change of the probability distribution $P_a(t)$. It is tempting to argue that the probability distribution

$$p_a(t) = \langle \psi | \mathbf{J}_a(t) | \psi \rangle \quad (2.8)$$

gives the probability that the event a happens between t and dt , since classically the probability that an event happened some time before time t is just the integral between some initial time t_o and t of the probability that the event happens at that time.

However, the probability distribution obtained from \mathbf{J}_a cannot be thought of as a probability distribution *in time*. $p_a(t)$ is not the probability that the event happened at time t . To see that $p_a(t)$ is not a probability distribution in time, let us compare its properties to the properties (1-4) of the conventional quantum¹ probability distribution obtained from the projectors Π_i .

Property 1 *The probability of finding that the system is in the state ϕ_i at time t is independent of the probability of finding that the system is in the state ϕ_j (at the same time t).*

¹properties 2-4 are also true of classical probability distributions

i.e.,

$$[\Pi_i(t), \Pi_j(t)] = 0. \quad (2.9)$$

If we interpret the probabilities $P_a(t)$ as probabilities in time, then our conventional notions of what these probabilities mean, break down. In general,

$$[\Pi_a(t), \Pi_a(t')] \neq 0. \quad (2.10)$$

Measurements made at earlier times influence measurements made at later times. The possible results of an observable at time t will depend on whether there were any previous measurements of that observable. In classical mechanics, one can make the interaction of the measuring device with the system arbitrarily weak, and therefore, not disturb the evolution of the system in time, but this is not true in quantum mechanics. A measurement of position at t_1 for example, will disturb the momentum of the particle in such a way that future measurements of position at t_2 will give very different results from the case where no measurement was performed at t_1 . Since $\Pi_a(t)$ does not commute with itself at different times, there is no reason to believe that \mathbf{J}_a will commute with itself at different times either. It is essentially this difference between conventional probabilities and those obtained from Π_a which prevents us from determining when an event occurred.

In addition, $p_a(t)$ and $P_a(t)$ do not have the following other properties of quantum

$$[\Pi_a(t), \Pi_a(t')] \neq 0 \quad \text{and} \quad [\Pi_a(t), \Pi_a(t')] \neq 0$$

different times. There is no reason why the event a can not happen at many different times. In general

$$\Pi_a(t)\Pi_a(t') \neq 0. \quad (2.12)$$

This is also true of classical distributions. For example, I can only be at one place at one time, but I can be at that same place at many different times.

Property 3 *The probabilities $P_i(t)$, are normalized at a given time.*

i.e.,

$$\sum_i P_i(t) = 1 \quad (2.13)$$

However, the operator \mathbf{J}_a is not necessarily normalized in time

$$\int_{-\infty}^{\infty} dt \frac{dP_a(t)}{dt} = \lim_{t \rightarrow \infty} P_a(t) - P_a(-t). \quad (2.14)$$

In special physical circumstances, $P_a(t)$ may initially be zero, and may finally equal one in the distant future, but there is no reason to expect this to be true in general. One might try to renormalize \mathbf{J}_a , but for each initial state the normalization will in general be different.

This property is also true of classical probability distributions. They also must be normalized, and the classical current is not always normalizable. For example, one can have many classical situations in which the event may never occur. The quantum case is more complicated however, since currents which are classically positive definite may be negative in the quantum case [29].

Lastly,

Property 4 *The probabilities $P_i(t)$ are positive definite.*

In general, $p_a(t)$ can be negative since $P(t)$ need not be monotonically increasing with time (this is obviously also true for classical probability functions). One can restrict \mathbf{J}_a

to only act on states for which \mathbf{P}_a is increasing with time, but the restricted domain of definition of \mathbf{J}_a may mean that it will no longer be self-adjoint. Furthermore, whether \mathbf{J}_a is positive or negative will not only depend on the state, but also on the Hamiltonian. For certain Hamiltonians, one may find that there are no states for which $p_a(t)$ does not take on negative values.

Another interesting aspect of \mathbf{J}_a and $\mathbf{\Pi}_a$ is that in general

$$[\mathbf{J}_a(t), \mathbf{\Pi}_a(t)] \neq 0. \quad (2.15)$$

The operator which measures that the event happened and the operator \mathbf{J}_a do not commute. If one believes that \mathbf{J}_a can be used to answer the question “when did the event happen?” then one finds that “when did it happen?” and “has it already happened?” seem to be complimentary (in Bohr’s sense) in that they interfere with each other. Naively, it would seem that determining “when did a occur?” would also answer the question “has a occurred?”. However the inaccuracy of the determination of “when did a occur?” seems to place limits on our ability to answer “has a occurred?”.

2.3 Time of a Measurement or Arrival

We now examine two specific examples of the determination of when an event occurred. In the first example, one tries to determine when a measurement occurred. In the second example, one wishes to determine the time at which a particle arrives to $x = 0$.

Let us try to measure the time that a measurement occurred (a measurement of a measurement in a sense). Imagine that we want to find out the time of a measurement of the observable \mathbf{A} of a quantum system S . The measurement of \mathbf{A} can be accomplished by coupling a macroscopic apparatus O to the system, via an *ergon* ~~gram~~ *ork*

here \mathbf{P} is the conjugate momentum to the pointer \mathbf{Q} of the measuring device, and $g(t)$ is a function which is zero everywhere, except during a small interval of time. After the measurement is complete, the measuring apparatus will be correlated with the state of the system. If initially, S is in a superposition of eigenstates $|\phi_i\rangle$ of the observable \mathbf{A} , so that $|\psi_S\rangle = \sum_i c_i |\phi_i\rangle$, then we expect the initial state of the combined $S - O$ system to evolve into a correlated state.

$$\sum_i c_i |\phi_i\rangle \otimes |O\rangle \rightarrow \sum_i c_i |\phi_i\rangle \otimes |O_i\rangle \quad (2.17)$$

here $|O\rangle$ is the original state of the device and the $|O_i\rangle$ are orthogonal states of the measuring apparatus which are correlated with the system. If the coupling is small, then the duration of the measurement might need to be long in order to distinguish between the various eigenvalues of \mathbf{A} . At any time during the measurement, it is possible to calculate the density matrix of the combined S-O system. One can imagine that a second apparatus O' measures the state of the first apparatus O to determine whether a measurement has occurred. This has been studied for the case when the measurement is gradual [21] [27]. Rovelli [28] has recently proposed that the apparatus O' might measure the operator

$$\mathbf{M} = \sum_i |\phi_i\rangle \otimes |O_i\rangle \langle O_i| \otimes \langle \phi_i|$$

In the case of time-of-arrival, one wishes to measure the time a particle arrives to a certain location (say $x = 0$). Often, the probability current is used to determine the arrival time[13]. One imagines that a particle is localized in the region $x < 0$ and traveling towards the origin. The projector

$$\mathbf{\Pi}_+ = \int_0^\infty dx |x\rangle\langle x| \quad (2.21)$$

is an operator which is equal to one when $x > 0$ and zero otherwise. The probability of detecting the particle in the positive x-axis is given by

$$P_+(t) = \langle \psi | \mathbf{\Pi}_+(t) | \psi \rangle. \quad (2.22)$$

In the Schrödinger representation, this expression is just $P_+(t) = \int_0^\infty |\psi(x, t)|^2 dx$. It is then claimed that the current \mathbf{J}_+ , given by

$$\frac{\partial \mathbf{J}_+}{\partial x} = \frac{d\mathbf{\Pi}_+(t)}{dt} \quad (2.23)$$

will give the probability that the particle arrives between t and $t + dt$.

It is clear that both the operators $\mathbf{M}(t)$ and $\mathbf{\Pi}_+(t)$ are specific examples of the operator $\mathbf{\Pi}_a$ discussed in Section 2.2. $\mathbf{M}(t)$ gives the probability at time t that a measurement has occurred. $\mathbf{\Pi}_+(t)$ gives the probability that the particle is found at $x > 0$ at time t . The two operators $\mathbf{m}(t)$ and $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ are examples of $\mathbf{J}_a(t)$. $\mathbf{m}(t)$ gives the change in the probability that the measurement happened at time t , while $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ gives the change in the probability that the particle is found at $x > 0$. However, one cannot interpret these operators as giving the probability that the measurement occurred (or the particle arrived). None of these operators allow one to measure the precise time at which the event occurred. They do not possess all the Properties 1-4 listed above.

Considered as probabilities in time, none of the operators above give distributions which have Property 1. The operators above do not commute with the Hamiltonian, and

therefore depend on t . For $t - t' \ll d\mathbf{H}$ we have for any operator $\mathbf{A}(t) \simeq \mathbf{A}(t') + i(t - t')[\mathbf{H}, \mathbf{A}(t')]$, and so

$$[\mathbf{A}(t), \mathbf{A}(t')] \simeq i(t - t')[\mathbf{H}, \mathbf{A}(t')], \mathbf{A}(t') \quad (2.24)$$

For arbitrary Hamiltonians, it is obvious that none of the operators above will commute with themselves at different times. Even for a free particle, one can explicitly calculate that neither $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ nor $\mathbf{\Pi}_+(t)$ commute with themselves at different times, (the calculation is neither difficult, nor particularly illuminating).

For some very specific states, and physical situations, Properties 2-4 may be obeyed, but this is certainly not true in general. For the case of time-of-arrival, even for a free Hamiltonian and wave packets which only contain modes of positive frequency, the current can be negative [29] (a violation of Property 4 - that probabilities must be positive definite). In fact, since the current is simply the time derivative of a projection operator, there is no reason to expect it to always be positive. For free particles which can arrive from the left and right, the current can be zero² and hence the probability distribution will be unnormalizable (Property 3). Also, in general, there is no reason why a particle can't be at the same position at many different times (a violation of Property 2). In the case of particles which move in a potential, one may find that there are no states for which Properties 3-4 are obeyed. For example, if there is an infinite potential barrier around the origin, the particle will never arrive, and the current will not be normalizable, and if there is a harmonic oscillator potential, the particle will cross the origin many times from both the left and the right violating Properties 2 and 4. For the case of determining when a measurement occurred, Rovelli restricts the class of measurements he considers to be those for which $\mathbf{m}(t)$ obeys Properties 2-4. As a result, $\mathbf{m}(t)$ cannot be used for arbitrary measurements. As with the time-of-arrival, there are clearly many Hamiltonians

²See Appendix A where we see that a coherent antisymmetric superposition of left and right moving waves has zero current.

for which Property 2-4 will be violated. Nor can $\mathbf{m}(t)$ be used for Hamiltonians for which its restricted domain of definition will mean that it is no longer self-adjoint.

Although operators such as \mathbf{m} and \mathbf{J}_+ do not commute with themselves at different times, it is possible to construct an operator which is time-translation invariant, and would give the time of an event in the classical limit. This will be discussed in Chapter 4 where we shall see that such an operator cannot be self-adjoint if the Hamiltonian is bounded from above or below.

2.4 Continual Event Monitoring

Instead of considering operators, a more physical meaningful method of measuring the occurrence of an event is to consider continuous measurement processes. For example, the operator $\mathbf{\Pi}_a(t)$ can be measured continuously or at small time intervals. When one considers such a physical measurement procedure one can see that the time at which an event occurs is not well defined in quantum mechanics. The probability of finding that the system enters one of the states ϕ_i at time t_a is given by the probability that it isn't in any of the states ϕ_i before t_a , times the probability that it is in one of the states ϕ_i at t_a .

To see how such a scheme might work, let us see how one could measure the time of an occurrence of the event corresponding to $\mathbf{\Pi}_a$. A measurement of the operator $\mathbf{\Pi}_a(t)$ will tell us whether the event has occurred at time t . We can then measure $\mathbf{\Pi}_a(t)$ at times $t_k = k\Delta$ for integral k in order to determine when the measurement occurred. Δ represents the frequency with which we monitor the system, and is therefore the inaccuracy of the measurement in time (it is the coarseness of the measurement in some sense).

We will not work in the Schrödinger representation, simply because it is the most

natural arena to talk about successive measurements on a system. At time t_1 , the probability that an event has occurred is given by

$$P(\uparrow, t_1) = \langle \psi_0(0) | \mathbf{U}_\Delta^\dagger \Pi_a \mathbf{U}_\Delta | \psi_0(0) \rangle \quad (2.25)$$

and the probability that it hasn't is

$$P(\downarrow, t_1) = \langle \psi_0(0) | \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta | \psi_0(0) \rangle \quad (2.26)$$

here \uparrow corresponds to detecting that the event has occurred, \downarrow corresponds to detecting that an event has not yet occurred, $\psi_0(0)$ is the initial state of the system and \mathbf{U}_Δ is the evolution operator $e^{-i\mathbf{H}\Delta}$. If the result is \downarrow , we collapse the wave function and evolve it to the next instant. The normalized state before the second measurement is:

$$|\psi_2(t_2)\rangle = \frac{\mathbf{U}_\Delta (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta |\psi_0\rangle}{\langle \psi_0 | \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta | \psi_0 \rangle^{1/2}} \quad (2.27)$$

The probability that an event has occurred at t_2 is given by the probability that an event didn't occur at t_1 times the probability that ψ_2 is in one of the states ϕ_i

$$P(\uparrow, t_2) = \frac{\langle \psi_0 | \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta^\dagger \Pi_a \mathbf{U}_\Delta (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta | \psi_0 \rangle}{\langle \psi_0 | \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta | \psi_0 \rangle} \times \langle \psi_0 | \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta | \psi_0 \rangle \quad (2.28)$$

The probability that an event didn't occur is given by

$$P(\downarrow, t_2) = \langle \psi_0 | \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta | \psi_0 \rangle \quad (2.29)$$

By repeating this process, we find that at time t_k the probability that an event has occurred is given by

$$P(\uparrow, t_k) = \langle \psi_0 | A_k | \psi_0 \rangle \quad (2.30)$$

here

$$A_k = \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta^\dagger (\mathbf{1} - \Pi_a) \dots \mathbf{U}_\Delta^\dagger \Pi_a \mathbf{U}_\Delta \dots (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta (\mathbf{1} - \Pi_a) \mathbf{U}_\Delta \quad (2.31)$$

and the probability that an event hasn't occurred is

$$P(\downarrow, t_k) = \langle \psi_0 | B_k | \psi_0 \rangle \quad (2.32)$$

with

$$B_k = \mathbf{U}_\Delta^\dagger (\mathbf{1} - \mathbf{\Pi}_a) \mathbf{U}_\Delta^\dagger (\mathbf{1} - \mathbf{\Pi}_a) \dots \mathbf{U}_\Delta^\dagger (\mathbf{1} - \mathbf{\Pi}_a) \mathbf{U}_\Delta \dots (\mathbf{1} - \mathbf{\Pi}_a) \mathbf{U}_\Delta (\mathbf{1} - \mathbf{\Pi}_a) \mathbf{U}_\Delta \quad (2.33)$$

By allowing the unitary operators to act on the projection operators we can write the A_k or B_k in the Heisenberg representation. For example

$$A_k = (\mathbf{1} - \mathbf{\Pi}_a)(t_1) \dots (\mathbf{1} - \mathbf{\Pi}_a)(t_{k-1}) \mathbf{\Pi}_a(t_k) (\mathbf{1} - \mathbf{\Pi}_a)(t_{k-1}) \dots (\mathbf{1} - \mathbf{\Pi}_a)(t_1) \quad (2.34)$$

However, while the operators $\mathbf{\Pi}_a(t)$ can be found by unitary time-evolution of $\mathbf{\Pi}_a(0)$, the operators A_k and B_k are not related by a unitary transformation to A_0 and B_0 . This already signals that they can not give an undisturbed distribution for the time of an event. Nor are the A_k and B_k projection operators.

The probabilities derived from A_k and B_k are not universal. In this case, they apply only to the specific measurement scenario under discussion. In particular the probability distribution is sensitive to the frequency at which $\mathbf{\Pi}_a$ is measured, a phenomenon which is related to the Zeno paradox [23].

As an example, consider a measurement of the spin of a particle. We wish to know at what time the measurement occurred. The particle is in a state given by

$$|\psi_S\rangle = a|\uparrow\rangle + b|\downarrow\rangle \quad (2.35)$$

and we use a simple measuring device which is also a spin 1/2 particle initially in the state $|O\rangle = |\uparrow'\rangle$, which evolves according to the Hamiltonian

$$\mathbf{H} = g(t) \sigma'_x \frac{1}{2} (1 - \sigma_z) \quad (2.36)$$

here $\int g(t)dt = \pi$ ($g(t)$ is sharply peaked, with width T), and the primed Pauli matrix acts on the measuring device, while the unprimed Pauli matrix acts on the system. After a time T , the spin of the measuring device will be correlated with the system. Since this measurement is rather crude, (the initial state of the device is the same as one of the measurement states), the operator \mathbf{M} at $t = 0$ is not zero. Let us simplify the problem further, by assuming that $a = 0$ and $b = 1$. In this case, the only relevant matrix element of $\mathbf{1} - \mathbf{M}$ is $|\downarrow\rangle \otimes |\uparrow'\rangle \langle \uparrow'| \otimes \langle \downarrow| = |\psi_o\rangle \langle \psi_o|$. We then find the probability that the measuring apparatus has not responded at time t_k is

$$\begin{aligned} P(\downarrow, t_k) &= |\langle \psi_o | U_\Delta | \psi_o \rangle|^{2k} \\ &\simeq |\langle \psi_o | 1 - i\Delta \mathbf{H} - \Delta^2 \mathbf{H}^2 | \psi_o \rangle|^{2k} \\ &\simeq 1 - \Delta^{2k} (\langle \mathbf{H}^2 \rangle - \langle \mathbf{H} \rangle^2)^k \end{aligned} \quad (2.37)$$

If we fix a value of $\tau = t_k$ and then make Δ go to zero, we find

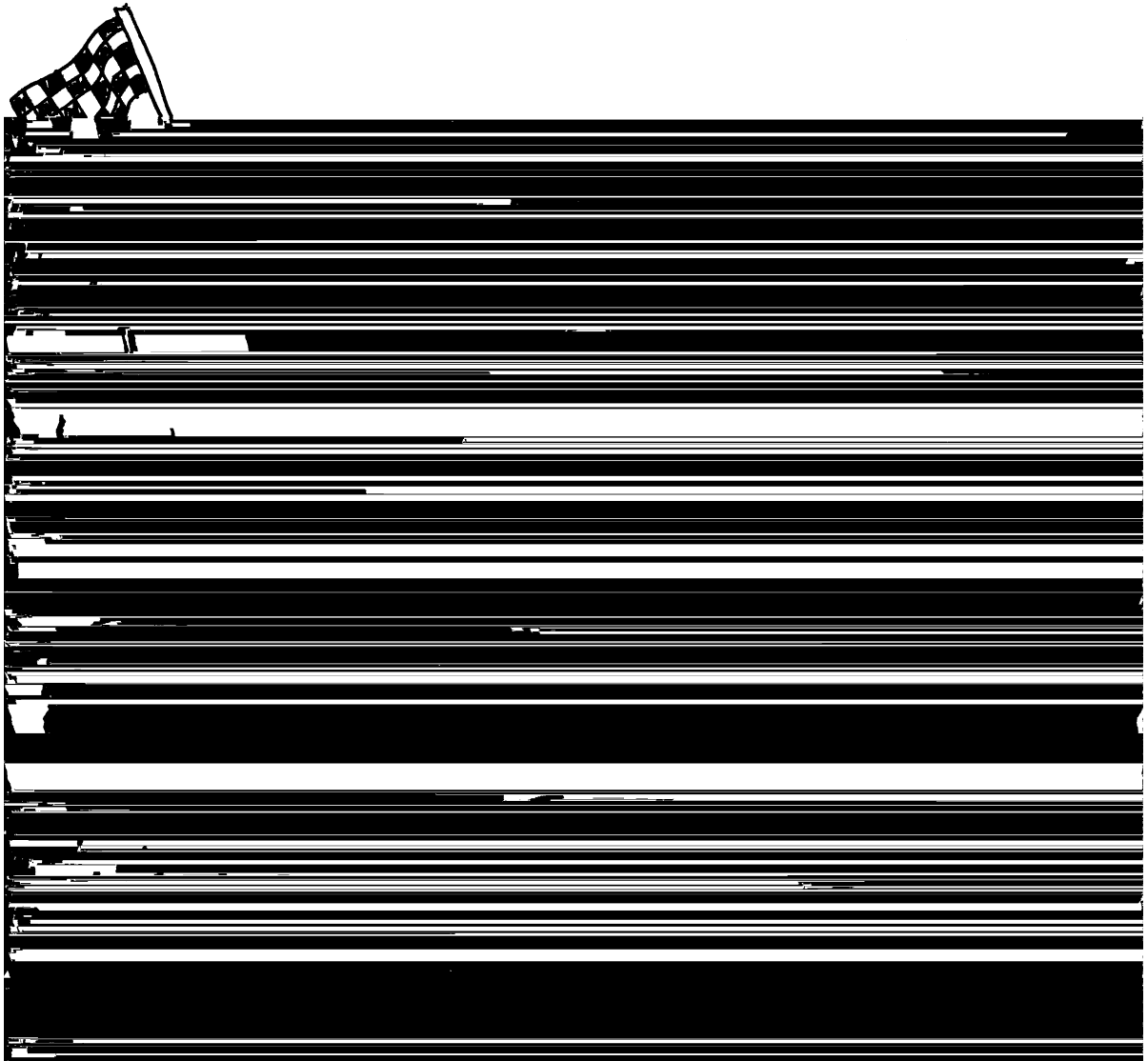
$$\begin{aligned} P(\downarrow, \tau) &\simeq e^{-(\Delta dE)^{2\tau/\Delta}} \\ &\simeq 1, \end{aligned} \quad (2.38)$$

which implies that the measuring apparatus becomes frozen and never records a measurement. In order not to freeze the apparatus, we need $\Delta > 1/dE$ (here dE is the uncertainty in energy of the measuring device O (initially the spacing between energy levels in this case)). There is always an inherent inaccuracy when measuring the time that the event (of the measurement) occurred. This inaccuracy is similar to the one which we will find in Chapter 3. Note that as discussed in the Introduction, this inaccuracy is not related to the so-called

One can of course use the set of operators A_k to compute a probability distribution in time, or experimentally determine a probability distribution for the time of an event. However, as we have just seen, this probability distribution is not a function of the system alone, but rather, it is related to the system and the measuring device (or set of operators). For example, the probability distribution will depend on Δ , and if Δ is too small, we will find that the event never occurs. The distribution $P(\uparrow, t_k)$ does allow us to predict the probabilities of future measurements.

Chapter 3

Physical Clocks and Time-of-Arrival



3.1 A Limitation on Time-of-Arrival Measurements

In the previous Chapter, we saw that if we attempt to measure the time of an event using a rapid series of measurements, then our measurements will disturb the very thing which we are trying to measure. For the simple case of a two state system, we were able to show that if we make the measurement accurately enough, the system freezes, and the event never occurs.

In this Chapter, we study measurements of the time-of-arrival of a particle to a particular location using physical clocks. The clocks are coupled to the system in such a way that when the particle arrives to the fixed location, the clock will read the time-of-arrival. Unlike the continuous measurement procedure discussed in the last Chapter, we obtain the time of the event using only a single measurement which is made well after the event has occurred. Nonetheless, we will find that if we make the measurement extremely accurate, the measurement will fail.

Consider a free particle, upon which a measurement is performed to determine the time-of-arrival to $x = x_A$. The time-of-arrival can be recorded by a clock situated at $x = x_A$ which switches off when the particle reaches it. In classical mechanics we could, in principle, achieve this with the smallest non-vanishing interaction between the particle and the clock, and hence measure the time-of-arrival with arbitrary accuracy.

In classical mechanics there are other indirect methods to measure the time-of-arrival. One could invert the equation of motion of the particle and obtain the time in terms of the location and momentum, $T_A(x(t), p(t), x_A)$. This function can be determined at *any time* t , either by a simultaneous measurement of $x(t)$ and $p(t)$ and evaluation of T_A , or by a direct coupling to $T_A(x(t), p(t), x_A)$. One could also measure the time-of-arrival using the method discussed in the previous chapter. By using a weak interaction which doesn't disturb the system, one can continually monitor the point of arrival to see if the particle

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From (3.41) we see that in order to use this clock to read the time, we need to know the initial position of the clock's dial $\mathbf{y}(t_0)$ and then subtract this from our final reading of \mathbf{y} . Quantum mechanics puts no limitation on how accurately this clock can be measured. If we want to accurately infer the time from the final reading of the clock then the clock must initially be prepared in a state with a very small uncertainty in y . At some later point, we can measure the coordinate $y(t_f)$ to any degree of accuracy we wish to infer the time from $y(t_f) - y(t_0)$.

of energy to turn them off. To measure the time-of-arrival of a particle, the particle itself will have to turn off the clock when it arrives – the external observer cannot supply any energy since she does not know when to turn the clock off. If the clock is much more energetic than the particle, then it will be impossible for the particle to turn off the clock and no record will exist that the particle arrived. In fact, **A**

that **B** does not know when the particle arrives. **B** can only know when the particle arrives if the clock is much less energetic than the particle. **B** can only know when the particle arrives if the clock is much less energetic than the particle.

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Hence at $x = x_{peak}$ the clock coordinate y is peaked at the classical time-of-arrival

$$y = \frac{mx_o}{k_0}. \quad (3.57)$$

To see that the clock yields a reasonable record of the time-of-arrival, let us consider further the probability distribution of the clock

$$\rho(y, y)_{x>0} = \int dx |\psi(x > 0, y, t)|^2. \quad (3.58)$$

In the case of inaccurate measurements with a small back-reaction on the particle $A_I \simeq 1$.

The clocks density matrix is then found (see Appendix B) to be given by:

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{y^2}{2\gamma(y)}}$$

Since the possible values obtained by p are of the order $1/\Delta_y \equiv 1/\Delta t_A$, the probability to trigger the clock remains of order one only if

$$\bar{E}_k \delta t_A > 1. \quad (3.61)$$

Here δt_A stands for the initial uncertainty in position of the dial \mathbf{y} of the clock, and is interpreted as the accuracy of the clock. \bar{E}_k can be taken as the typical initial kinetic energy of the particle.

In measurements with accuracy better than $1/\bar{E}_k$ the probability to succeed drops to zero like $\sqrt{E_k \delta t_A}$, and the time-of-arrival of most of the particles cannot be detected. Furthermore, the probability distribution of the fraction which has been detected depends on the accuracy δt_A and can become distorted with increased accuracy. This observation becomes apparent in the following simple example. Consider an initial wave packet that is composed of a superposition of two Gaussians centered around $k = k_1$ and $k = k_2 \gg k_1$.

trigger without including the clock:

$$H_{trigger} = \frac{1}{2m} \mathbf{P}_x^2 + \frac{\alpha}{2} (1 + \sigma_x) \delta(\mathbf{x}). \quad (3.62)$$

The particle interacts with the repulsive Dirac delta function potential at $x = 0$, only if the spin is in the $|\uparrow_x\rangle$ state, or with a vanishing potential if the state is $|\downarrow_x\rangle$. In the limit $\alpha \rightarrow \infty$ the potential becomes totally reflective (Alternatively, one could have considered a barrier of height α^2 and width $1/\alpha$.) In this limit, consider a state of an incoming particle and the trigger in the “on” state: $|\psi\rangle|\uparrow_z\rangle$. This state evolves to

$$|\psi\rangle|\uparrow_z\rangle \rightarrow \frac{1}{\sqrt{2}} \left[|\psi_R\rangle|\uparrow_x\rangle + |\psi_L\rangle|\downarrow_x\rangle \right], \quad (3.63)$$

here ψ_R and ψ_L are the reflected and transmitted wave functions of the particle, respectively.

The latter equation can be rewritten as

$$\frac{1}{2} |\uparrow_z\rangle (|\psi_R\rangle + |\psi_L\rangle) + \frac{1}{2} |\downarrow_z\rangle (|\psi_R\rangle - |\psi_L\rangle) \quad (3.64)$$

Since \uparrow_z denotes the “on” state of the trigger, and \downarrow_z denotes the “off” state, we have flipped the trigger from the “on” state to the “off” state with probability $1/2$ ². Although this model only works half the time, the chance of success does not depend in any way on the system, and in particular, on the particle’s energy. Furthermore, one can construct models where a detector is triggered almost all the time [35], although with some energy dependence in the probability of triggering.

So far we have succeeded in recording the event of arrival to a point. We have no information at all on the time-of-arrival. It is also worth noting that the net energy exchange between the trigger and the particle is zero, i.e., the particle’s energy is unchanged.

²It is interesting to see that for some wave functions which represent coherent superpositions of particles arriving from both the left and right, the detector is never triggered. An example of this is given in Appendix A.

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transmitted waves.

The eigenstates of (3.73), in the basis of σ_z , are given by

$$\Psi$$

There is however one limiting case in which the method does seem to succeed. Consider a narrow wave peaked around k with a width dk . To first order in dk , the probability T_{\downarrow} that the particle is successfully boosted is given by

$$T_{\downarrow} \simeq 1 - \frac{2dk}{k}. \quad (3.80)$$

Therefore in the special case that $\frac{dk}{k} \ll 1$, the transition probability is still close to one. If in this case we know in advance the value of k up to $dk \ll k$, we can indeed use the booster to improve the bound (3.61) on the accuracy.

The reason why this seems to work in this limiting case is as follows. The probability of flipping the particle's spin depends on how long it spends in the magnetic field described by the α term in (3.73). If however, we know beforehand, how long the particle will be in this field, then we can tune the strength of the magnetic field (α) so that the spin gets flipped. The requirement that $dk/k \ll 1$ is thus equivalent to having a small uncertainty in the "interaction time" with this field. In some sense, the measurement is possible, because we know the particle's momentum beforehand. Of course, if we have prior knowledge of the particle's momentum, then we could just measure \mathbf{x} and argue that this allows us to calculate the time of arrival through $t_A = mx/p$. We therefore believe that the reason the measurement procedure described above works in this limiting case is because it assumes prior knowledge of the particle's momentum, and we do not believe that one could improve it so that it works for all states. These "booster" measurements cannot be used for general wave functions, and even in the special case above, one still requires some prior information of the incoming wave function.

3.3.4 Gradual triggering of the clock

In order to avoid the reflection found in the previous two models, we shall now replace the sharp step-function interaction between the clock and particle by a more gradual

transition.

When the WKB condition is satisfied

$$\frac{d\lambda(x)}{dx} = \epsilon \ll 1 \quad (3.81)$$

here $\lambda(x)^{-2} = 2m[E_0 - V(x)]$, the reflection amplitude vanishes as

$$\sim \exp(-1/\epsilon^2) \quad (3.82)$$

Solving the equation for the potential with a given ϵ we obtain

$$V_\epsilon(x) = E_0 - \frac{1}{2m\epsilon^2} \frac{1}{x^2} \quad (3.83)$$

Note we observe that any particle with $E \geq E_0$ also satisfies the WKB condition (3.81) above for the *same* potential V_ϵ . Furthermore $p_y V_\epsilon$ also satisfies the condition for any $p_y > 1$.

These considerations suggest that we should replace the Hamiltonian in eq. (3.49)

with

$$H = \mathbf{P}_x^2/2m + V(x)\mathbf{P}_y \quad (3.84)$$

here

$$V(x) = \begin{cases} -\frac{x_A^2}{x^2} & x < x_A \\ -1 & x \geq x_A \end{cases} \quad (3.85)$$

Here $x_A^{-2} = 2m\epsilon^2$.

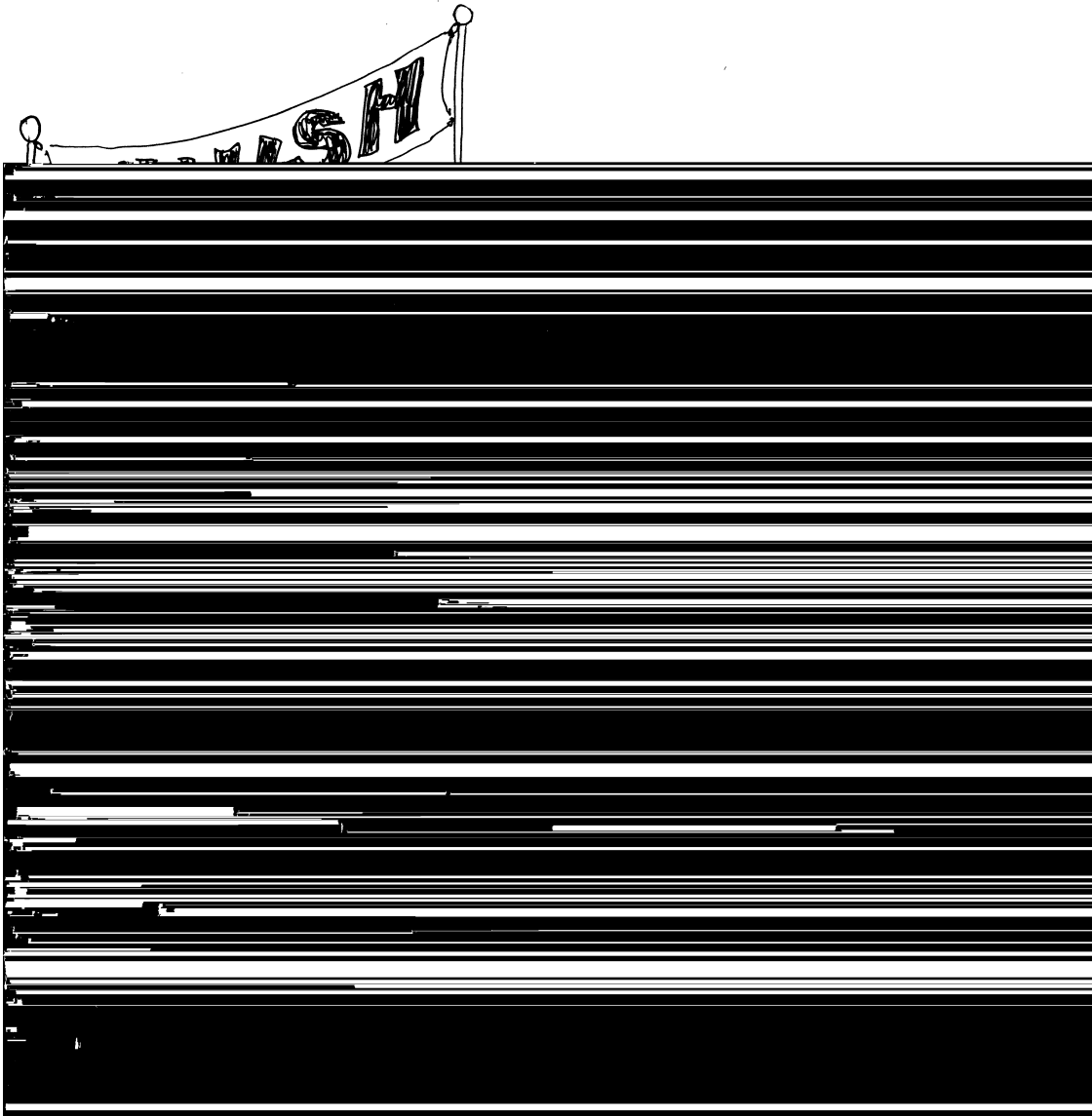
Thus this model describes a gradual triggering *on* of the clock which takes place when the particles propagates from $x \rightarrow -\infty$ towards $x = x_A$. In this case the arrival time is approximately given by $t - \mathbf{y}$ where $t = t_f - t_i$. Since without limiting the accuracy of the clock we can demand that $p_y \gg 1$, the reflection amplitude off the potential step is exponentially small for *any* initial kinetic energy E_k .

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Chapter 4

Time-of-Arrival Operators



4.1 Indirect Time-of-Arrival Measurements

In the previous chapter, we saw that one cannot measure the time-of-arrival of a free particle to arbitrary accuracy by coupling the particle to a clock. Still, one can imagine an indirect determination of arrival time by a measurement of some regularized time-of-arrival operator $\mathbf{T}(\mathbf{x}(t), \mathbf{p}(t), x_A)$ [9]. In quantum mechanics, ordinary observables like position and momentum *are* represented by operators at a fixed time t . However, we will show that there is no operator associated with the time it takes for a particle to arrive to a fixed location. In Section 4.2 we will prove formally that in general a Hermitian time-of-arrival operator with a continuous spectrum can only exist for systems with an unbounded Hamiltonian. This is because the existence of a time-of-arrival operator

the point of arrival with probability $1/2$. We also calculate the average energy of the states, in order to relate them to our proposal in Chapter 3 that one cannot measure the time-of-arrival to an accuracy better than $1/\bar{E}_k$. We end with concluding remarks in Section 4.7.

4.2 Conditions on A Time-of-Arrival Operator

As discussed in the previous section, although a direct measurement of the time-of-arrival may not be possible, one can still try to observe it indirectly by measuring some operator $\mathbf{T}(\mathbf{p}, \mathbf{x}, x_A)$. In the next two sections we shall examine this operator and its relation to

that equation (4.103) is inconsistent unless the Hamiltonian is unbounded from above and below [8].

4.3 Time-of-Arrival Operators vs. Continuous Measurements

Although formally there cannot exist a time-of-arrival operator \mathbf{T} , it may be possible to approximate \mathbf{T} to arbitrary accuracy [9]. This modified operator will be discussed more fully, in the next section, but for now, assume that we can define the regularized Hermitian operator

$$\mathbf{T}' = O(\mathbf{p})\mathbf{T}O(\mathbf{p}) \quad (4.104)$$

here $O(\mathbf{p})$ is a function which is equal to 1 at all values of p except around a small neighborhood of $k = 0$. For $|p| < \epsilon$, $O(\mathbf{p})$ goes rapidly to zero (at least as fast as \sqrt{k}). \mathbf{T}' is thus an operator which behaves just like \mathbf{T} except in a very small neighborhood

conjugate to H . The value of \mathbf{T}' is recorded on the conjugate of q – call it P_q . Now the uncertainty is given by $dT'_A = d(P_q) = 1/dq$, thus naively from $dq = 1/dT'_A < E_{min}$, we get $E_{min}dT' > 1$. However here, the average $\langle q \rangle$ is taken to be zero. There is no reason not to take $\langle q \rangle$ to be much larger than E_{min} , so that $\langle q \rangle - dq \gg -E_{min}$. If we do so, the measurement increases the energy of ψ and \mathbf{T}' is always conjugate to H . The limitation on the accuracy is in this case $dT'_A > 1/\langle q \rangle$ which can be made as small as we like.

However, even small deviations from the commutation relation (4.103) are problematic. Not only is the modification in \mathbf{T}' larger than E_{min} but also \mathbf{T}' is no longer conjugate to H .

state of the particle if the clock has stopped, and $|\chi'(t_A)\rangle$ the final state of the particle if the clock has not stopped.

Since the eigenstates of \mathbf{T} form a complete set, we can express any state of the particle as $|\psi\rangle = \int dt_A C(t_A) |t_A\rangle$. We then obtain :

$$\int dt_A C(t_A) |t_A\rangle |y = t_0\rangle \rightarrow \int dt_A C(t_A) |\chi(t_A)\rangle |y = t_A\rangle + \left(\int dt_A C(t_A) |\chi'(t_A)\rangle \right) |y = t\rangle. \quad (4.112)$$

The final probability to measure the time-of-arrival is hence $\int dt_a |C(t_a)\chi(t_a)|^2$. On the other hand we found that for a general wave function ψ , in the limit of $dt_a \rightarrow 0$, the probability for detection vanishes. Since the states of the clock, $|y = t_a\rangle$, are orthogonal in this limit, this implies that $\chi(t_a) = 0$ in eq. (4.111) for all t_a . Therefore, the eigenstates of \mathbf{T} cannot trigger the clock.

It should be mentioned however, that one way of circumventing this difficulty may be to consider a coherent set of \mathbf{T} eigenstates instead of the eigenstates themselves. These normalizable states will no longer be orthogonal to each, so they may trigger the clock if they have sufficient energy¹. In this regard it is of interest to prematurely quote a result which we will show in Section 4.5 - the average energy of a Gaussian distribution of time-of-arrival eigenstates is proportional to $1/\Delta$ where Δ is the spread of the Gaussian. This puts us at the edge of the limitation given in Equation (3.91).

¹An arbitrary wave packet can be written as a superposition of normalized eigenstates, and yet we know that arbitrary wave packets do not trigger the clocks of Section 3.3.1. This creates a somewhat interesting situation if normalized eigenstates trigger a clock, but wave packets made of superpositions of them do not.

4.4 The Modified Time-of-Arrival Operator

Kinematically, one expects that the time-of-arrival operator for a free particle arriving at the location $x_A = 0$ might be given by

$$\mathbf{T} = -\frac{m}{2} \frac{1}{\sqrt{\mathbf{p}}} \mathbf{x}(0) \frac{1}{\sqrt{\mathbf{p}}}. \quad (4.113)$$

The operator $-m(\frac{1}{\mathbf{p}}\mathbf{x} + \mathbf{x}\frac{1}{\mathbf{p}})$ is equivalent to the one above as can be seen by use of the commutation relations for \mathbf{x} and \mathbf{p} .

In the k representation this operator can be written as

$$\mathbf{T}(k) = -im \frac{1}{\sqrt{k}} \frac{d}{dk} \frac{1}{\sqrt{k}} = -im \left(\frac{1}{k} \frac{d}{dk} + \frac{d}{dk} \frac{1}{k} \right) \quad (4.114)$$

here $\sqrt{k} = i\sqrt{|k|}$ for $k < 0$. If one solves the eigenvalue equation, one finds a set of anti-symmetric states ² for this operator given by

$$g_{t_A}^a(k) = (\theta(k) - \theta(-k)) \frac{1}{\sqrt{2\pi m}} \sqrt{|k|} e^{i\frac{t_A k^2}{2m}}. \quad (4.115)$$

The symmetric states are

$$g_{t_A}^s(k) = (\theta(k) + \theta(-k)) \frac{1}{\sqrt{2\pi m}} \sqrt{|k|} e^{i\frac{t_A k^2}{2m}} \quad (4.116)$$

However, the operator is not self-adjoint and these states are not orthogonal.

$$\langle t'_A | t_A \rangle = \frac{1}{2\pi m} \int_0^\infty dk^2 e^{\frac{i}{2m} k^2 (t_A - t'_A)} = \delta(t_A - t'_A) -$$

Trying to make \mathbf{T} self-adjoint by defining boundary conditions at $k = 0$ leads to the requirement on square integrable wave functions $u(k), v(k)$ such that

$$\langle u, \mathbf{T}v \rangle - \langle \mathbf{T}^*u, v \rangle = im \left[\lim_{k \rightarrow 0^-} \frac{v(k)\overline{u(k)}}{|k|} + \lim_{k \rightarrow 0^+} \frac{v(k)\overline{u(k)}}{|k|} \right] = 0 \quad (4.118)$$

i.e., the boundary conditions must be chosen so that $\frac{v(k)\overline{u(k)}}{k}$ is continuous through $k = 0$. This continuity condition cannot force $u(k)$ to have the same boundary conditions as $v(k)$ for any choice of boundary condition on $v(k)$. For example, if we choose $v(k)/\sqrt{k}$ to be continuous through the origin, then $u(k)/\sqrt{k}$ must be anti-continuous through the origin. I.e. the domain of definition of \mathbf{T} and \mathbf{T}^* differ and \mathbf{T} cannot be self-adjoint. This is not at all surprising, given the proof in Section 4.2.

One might however, try to modify \mathbf{T} in order to make it self-adjoint in the manner shown in Section 4.3 [19]. Consider the operator

$$\mathbf{T}_\epsilon(k) = -im\sqrt{f_\epsilon(k)} \frac{d}{dk} \sqrt{f_\epsilon(k)}$$

where $f_\epsilon(k)$ is some smooth function which differs from $1/k$ only near $k = 0$. Since $v(k)$ could diverge at the origin at a rate approaching $1/\sqrt{k}$ and still remain square-integrable, if $f_\epsilon(k)$ goes to zero at least as fast as k , then \mathbf{T}_ϵ will be self-adjoint and defined over all square integrable functions. However, as we show in Sections 4.5 and 4.6, these eigenstates do not behave as one would expect a time of arrival eigenstate to behave.

It can be verified that \mathbf{T}_ϵ has a degenerate set of eigenstates $|t_A, +\rangle$ for $k > t_A$, $|t_A, -\rangle$ for $k < 0$, given by

states given by

$$f_\epsilon(k) = \begin{cases} \frac{k}{\epsilon^2} & |k| < \epsilon \\ \frac{1}{k} & |k| > \epsilon \end{cases} \quad (4.121)$$

If $\epsilon \rightarrow 0$, one might expect \mathbf{T}_ϵ

Its modulus squared vanishes when integrated around a small neighborhood of $x = 0$. ${}_{\epsilon}\tilde{g}^+(x)_{t_A}$ then, is not localized around the point of arrival, at the time-of-arrival. This will also be verified in the next section where we examine normalizable states. Although ${}_{\epsilon}\tilde{g}^+(x)_{t_A}$ is not localized around the point of arrival at the time of arrival, one might hope that this part of the state does not contribute significantly in time-of-arrival measurements when $\epsilon \rightarrow$

As argued in the previous section, the second term should act like a time-of-arrival state. The first term is due to the modification of \mathbf{T} and has nothing to do with the time of arrival. We will first show that the second term can indeed be localized at the point-of-arrival $x = 0$ at the time of arrival $t = t_A$. We will do this by expanding it around $x = 0$ in a Taylor series. After taking the limit $\epsilon \rightarrow 1$ \mathcal{U}

here

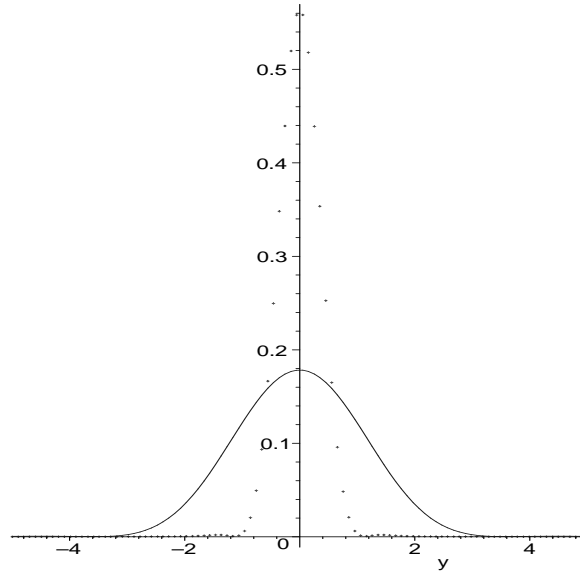
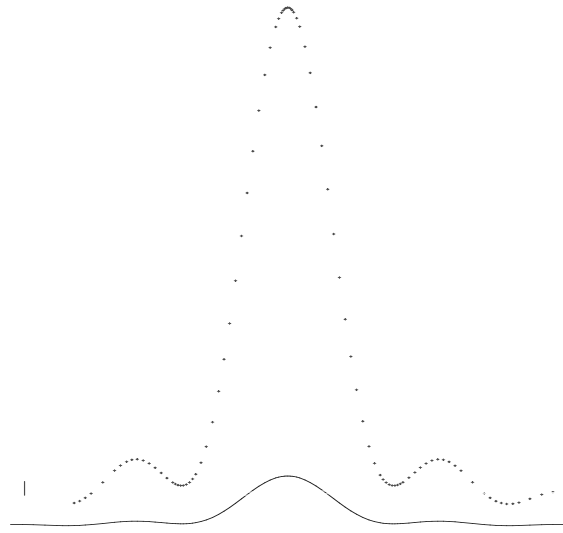


Figure 4.1: Unmodified part of time-of-arrival eigenstate. $|_o\tau^+(x, \tau)|^2$ vs. x , with $\Delta = m$ (solid line), and $\Delta = \frac{m}{10}$ (dashed line). As Δ gets smaller, the probability function gets more and more peaked around the origin.

If $i\epsilon x$ is not large, we can use the fact that for Δ and ϵ very small, $i\epsilon^2 t_A/m \ll 1/2$ so that we have

$$_o\tau^+(x, 0) \simeq (2\pi)^{\frac{1}{4}} \sqrt{\frac{\epsilon^3 \Delta}{2m}} \frac{\Phi(\sqrt{-i\epsilon x})}{\sqrt{-i\epsilon x}}$$



Chapter 4. Time-of-Arrival Operator

As ϵ or Δ go to zero, N_ϵ diverges, and if we renormalize the state, the entire norm will be made up of the modified part of the eigenstate.

It is also of interest to calculate the average value of the kinetic energy for these states, in order to see whether these states will trigger the physical clocks discussed in Chapter 3. In calculating the average energy, the modified piece will not matter since k^2 goes to zero at $k = 0$ faster than $\frac{1}{\sqrt{k}}$ diverges. We find

$$\begin{aligned} \langle \tau_\Delta^+ | \mathbf{H}_k | \tau_\Delta^+ \rangle &= \int dk \frac{k^2}{2m} \langle \tau_\Delta^+ | k \rangle \langle k | \tau_\Delta^+ \rangle \\ &= \frac{N^2}{\pi(2m)^2} \int_0^\infty k^3 e^{\frac{i(t_A - t'_A)k^2}{2m}} e^{-\frac{t_A^2 + t'^2_A}{\Delta^2}} dt_A dt'_A dk \\ &= \frac{4}{\Delta\sqrt{2\pi}} \end{aligned} \tag{4.142}$$

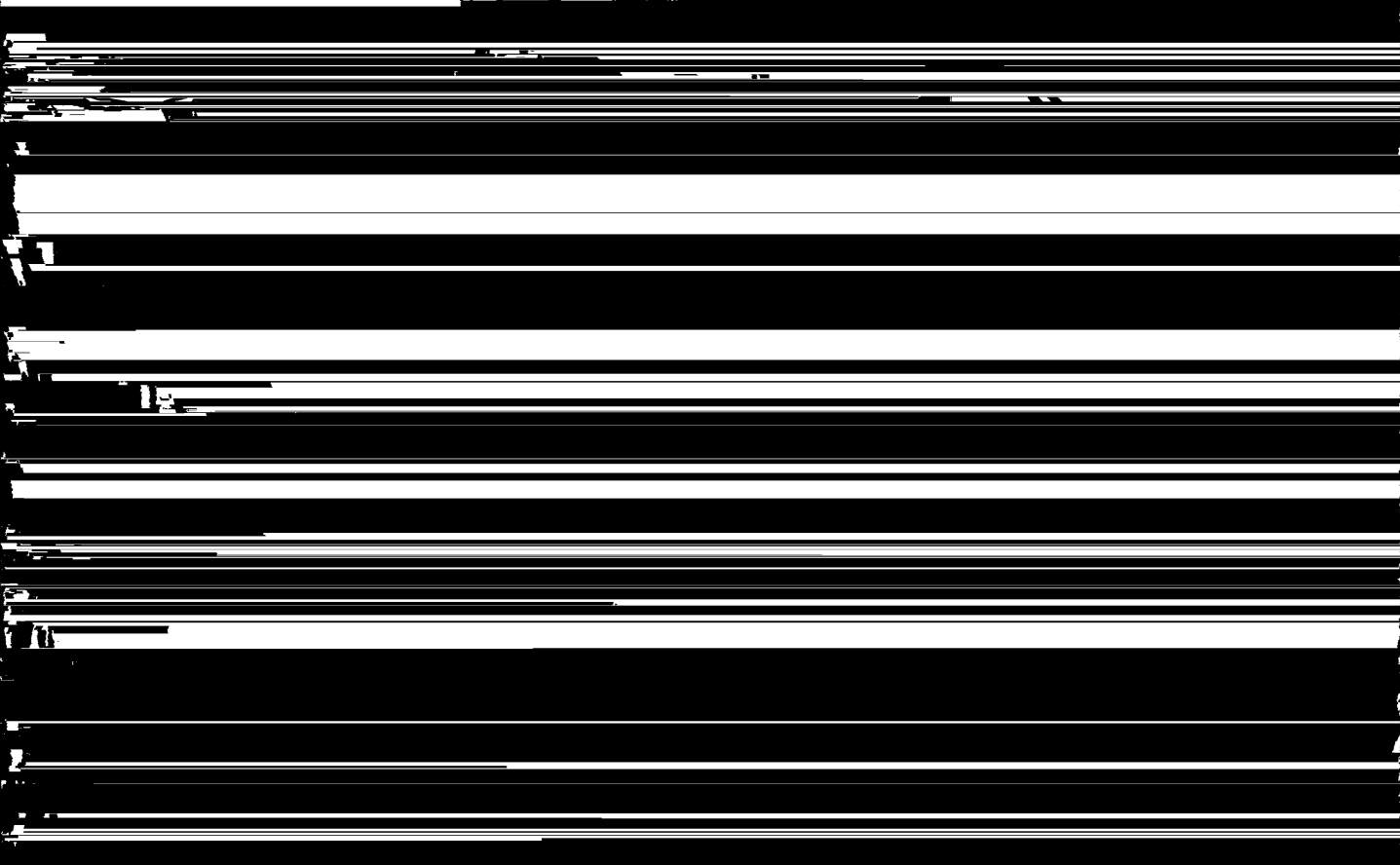
We see therefore, that the kinematic spread in arrival times of these states is proportional to $1/\bar{E}_k$. Since the probability of triggering the model clocks discussed in Chapter 3 decays as $\sqrt{E_k \delta t_A}$, here δt_A is the accuracy of the clock, we find that the states $|\tau_\Delta^+\rangle$ will not always trigger a clock whose accuracy is $\delta t_A = \Delta$.

4.7 Limited Physical Meaning of a Time-of-Arrival Operator

We would also like to stress that continuous measurements differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. Operationally one performs here two completely different measurements. While the time-of-arrival operator is a formally constructed operator which can be measured by an impulsive von-Neumann interaction, it seems that continuous measurements are much more closer to actual experimental set-ups. Furthermore, we have seen that the result of these two measurements do not need to agree, in particular in the high accuracy limit, continuous measurements give rise to entirely different behavior. This suggests that as in the case of the problem of finding a “time operator” [20] for closed quantum systems, the time-of-arrival operator has a somewhat limited physical meaning.

Chapter 5

Traversal Time



5.1 A Limitation on Traversal Time Measurements

5.2 Measuring Momentum Through Traversal-Distance

The measurement of traversal-distance may be considered the space-time *dual* of the measurement of traversal time: instead of fixing x_1 and x_2 and measuring $t_F = t_2 - t_1$, one fixes t_1 and t_2 and measures $x_F = x_2 - x_1$. It is instructive to examine first this simpler case of traversal-distance and point out the similarities and the differences.

Unlike the case of traversal time, a measurement of traversal-distance can be described by the standard von Neumann interaction. For a free particle the Hamiltonian is

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \mathbf{Q}\mathbf{x} \left[\delta(t - t_1) - \delta(t - t_2) \right] \quad (5.144)$$

here \mathbf{Q} is the coordinate conjugate to the pointer variable \mathbf{P} . The change in \mathbf{P} yields the traversal-distance:

$$\mathbf{P}(t > t_2) - \mathbf{P}_0 = \mathbf{x}(t_2) - \mathbf{x}(t_1) = \mathbf{x}_F. \quad (5.145)$$

However the measurement of the traversal-distance provides additional information: it also determines the momentum \mathbf{p} of the particle *during* the time interval $t_1 < t < t_2$. From the equations of motion at T

Chapter 5.

It is also commonly accepted that the dwell time operator [31], given by

$$\tau_{\mathbf{D}} = \int_0^{\infty} dt \mathbf{\Pi}_{x_A}(t) \quad (5.150)$$

here

$$\mathbf{\Pi}_{x_A}(0) = \int_{x_1}^{x_2} |x\rangle\langle x| \quad (5.151)$$

can be used to compute the traversal time¹. Such a quantity however, cannot be measured, since, as we saw in Chapter 2, the operator $\mathbf{\Pi}_{x_A}(t)$ does not commute with itself at different times.

$$[\mathbf{\Pi}_{x_A}(t), \mathbf{\Pi}_{x_A}(t')] \neq 0. \quad (5.152)$$

Therefore, one must measure the traversal time in a more physical way. One must demand that if we measure the traversal time to be t_F , then the particle must actually traverse the distance between x_1 and x_2 in the time given by the traversal time measurement. For example, one could have a clock which runs when the particle is between x_1 and x_2 given by the Hamiltonian [16][32]

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\mathbf{Q} \quad (5.153)$$

where the traversal time is given by the variable \mathbf{P} conjugate to \mathbf{Q} and the potential V is equal to 1 when $x_1 \leq x \leq x_2$ and 0 everywhere else. In the Heisenberg picture, the equations of motion are

$$\dot{\mathbf{x}} = \mathbf{p}/m, \quad \dot{\mathbf{p}} = -\mathbf{Q}(\delta(\mathbf{x} - x_1) - \delta(\mathbf{x} - x_2)) \quad (5.154)$$

$$\dot{\mathbf{P}} = V(\mathbf{x}), \quad \dot{\mathbf{Q}} = 0. \quad (5.155)$$

The particle's momentum is disturbed during the measurement

$$\mathbf{p}' = \sqrt{\mathbf{p}^2 - 2m\mathbf{Q}} \quad (5.156)$$

¹in our case, where there is no potential barrier, the dwell time and traversal time are equivalent

here \mathbf{p}'

here the particle has gone through the detector $P = t_F = \frac{mL}{p_o}$. We therefore need the condition

$$dP < \frac{mL}{p_o}. \quad (5.162)$$

Since at best we have $dP = 1/dQ$, we find

$$dp'dx = dp'L > 1. \quad (5.163)$$

The uncertainty relation (5.158) only applies to this particular model clock - it might be possible to accurately measure the traversal time in some clever way. In the following section we will argue that this cannot be done, by demonstrating that this uncertainty applies to all measurements of traversal time.

Finally, we should note that a traversal time detector could be made by measuring the time-of-arrival to x_1 and the time-of-arrival to x_2 . This would require two time-of-arrival clocks each with its own inaccuracy, whereas the model above only has one clock.

5.4 General Argument for a Minimum Inaccuracy

We now consider general measurements of traversal time. We will however, impose some fairly unrestrictive requirements on the measurement. We will assume that the measurement does not prevent the particle from actually traversing the distance between x_1 and x_2 . I.e. we want to be able to say that the particle did indeed traverse the distance L - otherwise, it is unclear what it is that we are measuring.

We also demand that the result of the measurement corresponds in some sense with the classical notion of traversal time. I.e., we are measuring something like mL/p . Our measuring device will consist of a pointer, which is set to some initial position P_o with an uncertainty in the initial value of the pointer of dP . At the end of the measurement, we assume that the value of the traversal time is inferred accurately by reading the final

value of the pointer. The measured traversal time is then proportional to $P_f - P_i$. The relative accuracy of the traversal time will then be given by $\delta T_f/T_f - dP/(P_f - P_i)$

Another condition we will impose is that the inaccuracy of the measurement, δT_F , is less than the quantity we are trying to measure \mathbf{T}_F (i.e. we are looking at accurate measurements). Finally, we assume that the experimentalist has no knowledge of the state of the particle, and thus must set the initial state of the measuring device (and its inaccuracy δP) with no prior knowledge of the ensemble.

Before proceeding with the argument, we should be clear to distinguish between different types of uncertainties. For an operator \mathbf{A} , there exists a kinematic uncertainty which we will denote by $d\mathbf{A}$ given by

$$d\mathbf{A}^2 = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2. \quad (5.164)$$

This is the uncertainty in the distribution of the observable A . There is also the inherent *inaccuracy* in the measuring device. This is the relevant quantity in equations (5.143) and (5.158). It refers to the uncertainty in the initial state of the measuring device's pointer position P , and we will denote it by δA . For our measuring devices we have

$$\delta A = dP_o \quad (5.165)$$

This inaccuracy applies to each individual measurement. Lastly, there is the uncertainty ΔA which applies to the spread in measurements made on the ensemble. Given a set A_M of experiments $i = 1, 2, 3, \dots$ which yield results A_i , we have

$$\Delta A^2 = \langle A_M^2 \rangle - \langle A_M \rangle^2. \quad (5.166)$$

This uncertainty includes a component due to the kinematic uncertainty of the attribute of the system, and also the inaccuracy of the device. For our measuring device, the kinematic spread in the pointer position at the end of each experiment gives ΔA

$$\Delta A = dP_f \quad (5.167)$$

The Heisenberg uncertainty relationship $dA dB > 1$ applies to measurements on ensembles. Given an ensemble, we measure **A** on half the ensemble and **B** on the other half. The uncertainty relation also applies to simultaneous measurements². If we measure **A** and **B** simultaneously on each system in the ensemble, then the distributions of **A** and **B** must still satisfy the uncertainty relationship.

Returning now to the traversal time, we see that it can be interpreted as a simultaneous measurement of position and momentum. We know the particle's momentum p during the time that it passes between $x = x_1$ and $x = x_2$ from the classical equations of motion

$$t_F = \frac{mL}{p}. \quad (5.168)$$

In other words, eigenstates of momentum must have traversal times given by equation (5.168). This measurement of momentum is analogous to the measurement described in



the particle and so a measurement of \mathbf{X} is also a measurement of \mathbf{x} . This is what we mean by a local interaction.

and does not depend on the nature of the ensemble upon which we will be making measurements. For a free Hamiltonian, a measurement of the traversal time will result in a final pointer position given by

$$P_f = P_o + \frac{mL}{p} \quad (5.172)$$

here p is the momentum of the particle in the absence of any measuring device. For eigenstates of \mathbf{p} (or states peaked highly in p), we demand that the traversal time be given by the classically expected value³. Recall that the kinematic spread in the particle's momentum is given by $d\mathbf{p}^2 = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2$. A measurement of the traversal time for a particular experiment i can take on the values

$$\begin{aligned} T_i &= P_f \\ &= \frac{mL}{p} + \epsilon \end{aligned} \quad (5.173)$$

A given measurement T_i will allow us to infer the momentum of the particle p_i during the measurement

$$p_i(T_i) = \frac{mL}{T_i} = \frac{mLp}{mL + p\epsilon}. \quad (5.174)$$

The average value of any pointer α of the measured momentum is

$$\langle p_M \rangle = \int \left(\frac{mLp}{mL + p\epsilon} \right) f(p)g(\epsilon)dpd\epsilon \quad (5.175)$$

here $f(p)$ gives the distribution of the particle's momentum and $g(\epsilon)$ is the distribution of the fluctuations. We now choose the mass m of the ensemble so that we always have

$$\epsilon p \ll mL. \quad (5.176)$$

In other words, we consider measurements on ensembles where the measurement is much more accurate than the quantity being measured. i.e.. $\delta T_F \ll T_F$. Indeed for the

³It is possible to include small deviations from the classical value, by including an additional term in (5.172). These fluctuations need to average to zero in order to satisfy the correspondence principle. For small fluctuations, the following discussion is not altered.

example given in the previous section, for every given ϵ and p

then implies

$$\delta T_F^2 > \frac{1 - \frac{1}{4}L^2 dp^2}{\langle E \rangle^2 + dE^2}. \quad (5.186)$$

Note that we can arrange our experiment with Ldp arbitrarily small, by choosing dp im

Chapter 6

Order of Events

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6.1 Past and Future

The notion that events proceed in a well defined sequence is unquestionable in classical

If the state evolves irreversibly to a state for which $\Pi_A \Psi(t) = 1$, then we can easily measure whether the event A has occurred at any time t . We could therefore measure whether a free particle arrives to a given location before or after a classical time t_B . Of course, for many systems, the system will not irreversibly evolve to the required state. For example, a particle influenced by a potential may cross over the origin many times. However, for an event such as atomic decay, the probability of the atom being re-excited is relatively small, and one can argue that the event is more or less irreversible.

For the case of a free particle which is traveling towards the origin from $x < 0$ one can argue that if at a later time I measure the projection operator onto the positive axis and find it there, then the particle must have arrived to the origin at some earlier time. This is in some sense a definition, because we know of no way to measure the particle being at the origin without altering its evolution or being extremely lucky and happening to measure the particle's location when it is at the origin.

While measuring whether an event happened before or after a fixed time t_B may be possible, we will find that for two quantum systems, one cannot in general measure whether the time t_A of event A , occurred before or after the time t_B of event B .

In Section 6.2, confining ourselves to a particular example of order of events, we will consider the question of order of arrival in quantum mechanics. Given two particles, can

we be able to measure the order of arrival in quantum mechanics? In our earlier discussion of the measurement problem, we saw that the measurement process is irreversible and that the system collapses to a definite state. This suggests that the measurement process is irreversible and that the system collapses to a definite state. This suggests that the measurement process is irreversible and that the system collapses to a definite state.

the order of arrival by measuring the operator

$$\mathbf{O} = \text{sgn}(\mathbf{T}_x - \mathbf{T}_y) \quad (6.189)$$

here T_x and T_y are the time-of-arrival operators associated with each particle.

In Section 6.3 we discuss measurements of coincidence. I.e., can we determine whether both particles arrived at the same time. Such measurements allow us to change the accuracy of the device before each experiment. We find that the measurement fails when the accuracy is made better than $1/\bar{E}$.

In Section 6.4 we discuss the relationship between ordering of events and the resolving power of Heisenberg's microscope, and argue that in general, one cannot prepare a two-particle state which is always coincident to within a time of $1/\bar{E}$.

6.2 Which first?

We now examine a case where the time t_B is not given by a classical clock, but rather a quantum system. Consider two free particles (which we will label as x and y) initially localized to the right of the origin, and traveling to the left. We then ask whether one can measure which particle arrives to the origin first. The Hamiltonian for the system and measuring apparatus is given by

$$\mathbf{H} = \frac{\mathbf{P}_x}{2m} + \frac{\mathbf{P}_y}{2m} + \mathbf{H}_i \quad (6.190)$$

here \mathbf{H}_i is some interaction Hamiltonian. For example, a promising interaction Hamiltonian is

$$\mathbf{H}_i = \alpha \delta(\mathbf{x}) \theta(-\mathbf{y}) \quad (6.191)$$

with α going to infinity. If the y -particle arrives before the x -particle, then the x -particle will be reflected back. If the y -particle arrives after the x -particle, then neither particle

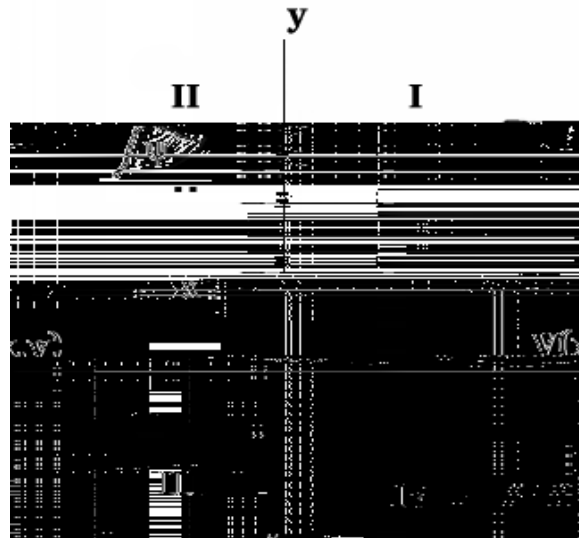


Figure 6.3: A potential which can be used to measure which of two particles came first (given by $V(x, y) = \alpha\delta(\mathbf{x})\theta(-\mathbf{y})$). The wave function for two incoming particles in one dimension looks like a single wave packet in two dimensions travelling towards the origin.

sees the potential, and both particles will continue traveling past the origin. One can therefore wait a sufficiently long period of time, and measure where the two particles are. If both the x and y particles are found past the origin, then we know that the x-particle arrived first. If the y-particle is found past the origin while the x-particle has been reflected back into the positive x-axis then we know that the y-particle arrived first.

Classically, this method would appear to unambiguously measure which of the two particles arrived first. However, in quantum mechanics, this method fails. From (6.190) we can see that the problem of measuring which particle arrives first is equivalent to deciding where a single particle traveling in a plane arrives. Two particles localized to the right of the origin is equivalent to a single particle localized in the first quadrant (see Figure 6.3). The question of which particle arrives first, becomes equivalent to the

to the origin (the sharp edge of the potential). The amplitude for being scattered off the region around the edge in the direction θ is given by $|f(r, \theta)|^2$.

It might be argued that since these particles scattered, they must have scattered off the potential, and therefore they represent experiments in which the y-particle arrived first. However, this would clearly over count the cases where the y-particle arrived first. We could have just as easily have placed our potential on the negative x-axis, in which case, we would over count the cases where the x-particle arrived first.

In the “interference region” we cannot have confidence that our measurement worked at all. We should therefore define a “failure cross section” given by

$$\begin{aligned}\sigma_f &= \int_0^{2\pi} |f(\theta)|^2 \\ &= \frac{1}{k \cos(\frac{\theta_0}{2})}\end{aligned}\tag{6.195}$$

From (6.195) we can see that cross section for scattering off the edge is the size of the particle’s wavelength multiplied by some angular dependence. Therefore, if the particle arrives within a distance of the origin given by

$$\delta x > 2/k\tag{6.196}$$

the measurement fails. We have dropped the angular dependence from (6.195) – the angular dependence is not of physical importance for measuring which particle came first, as it depends on the details of the potential (boundary conditions) being used. The particular potential we have chosen is not symmetrical in x and y. From this we can

In other words, our measurement procedure relies on making an inference between time measurements and spatial coordinates. The last two equations then give us

$$\delta t > \frac{1}{E} . \quad (6.198)$$

One will not be able to determine which particle arrived first, if they arrive within a time $1/E$ of each other, where E is the total kinetic energy of both particles. Note that Equation (6.198) is valid for a plane wave with definite momentum k . For wave functions for which $dk \ll k$, one can replace E by the expectation value $\langle E \rangle$. However, for wave functions which have a large spread in momentum, or which have a number of distinct peaks in k , then to ensure that the measurement almost always works, one must measure the order of arrival with an accuracy given by

$$\delta t > \frac{1}{\bar{E}} \quad (6.199)$$

where \bar{E} is the minimum typical total energy¹.

Although it seemed plausible that one could measure which particle arrived first, we found that if the particles are coincident to within $1/\bar{E}$, then the measurement fails.

6.3 Coincidence

In the previous model for measuring which particle arrived first, we found that if the two particles arrived to within $1/\bar{E}$ of each other, the measurement did not succeed. The width $1/\bar{E}$ as an inherent inaccuracy which could not be overcome. However, in our simple model, we were not able to adjust the accuracy of the measurement.

It is therefore instructive to consider a measurement of “coincidence” alone for which one can quite naturally adjust the accuracy of the experiment. Given two particles

¹For example, one need not be concerned with exponentially small tails in momentum space, since the contribution of this part of the wave function to the probability distribution will be small. If however, $\psi(E)$ has two large peaks at E_{small} and E_{big} spread far apart, then if δt does not satisfy $\delta t > 1/E_{small}$ one will get a distorted probability distribution. For a discussion of this, see Chapter 3.

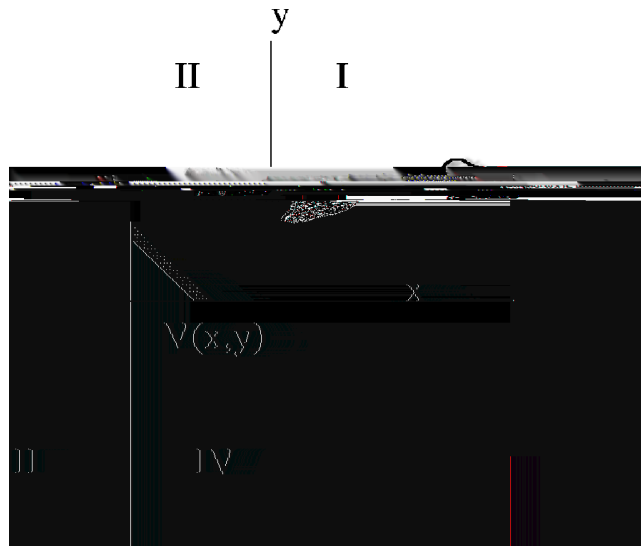


Figure 6.4: Potential for measuring whether two particles are coincident.

traveling towards the origin, we ask whether they arrive within a time δt_c of each other. If the particles do not arrive coincidentally, then we do not concern ourselves with which arrived first. The parameter δt_c can be adjusted, depending on how accurate we want our coincident “

mechanically, we once again find an interference region around the strip which scatters particles into the classically forbidden regions of quadrant two and four. The shadow is not sharp, and we are not always certain whether the particles were coincident.

A solution to plane waves scattering off a narrow strip is well known and can be found in many quantum mechanical texts (see for example [46] where the scattered wave is written as a sum of products of Hermite polynomials and Mathieu functions). However, for our purposes, we will find it convenient to consider a simpler model for measuring coincidence, namely, an infinite circular potential of radius a , centered at the origin.

$$H_i = \alpha V(r/a) \quad (6.200)$$

where $V(x)$ is the unit disk, and we take the limit $\alpha \rightarrow \infty$.

It is well known that if $a < 1/k$, then there will not be a well-defined shadow behind the disk. To see this, consider a plane wave coming in from negative x -infinity. It can be expanded in terms of the Bessel function $J_m(kr)$ and then written asymptotically ($r \gg 1$) as a sum of incoming and outgoing circular waves.

$$\begin{aligned} e^{ikx} &= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta \\ &\simeq \sqrt{\frac{1}{2\pi ikr}} \left[e^{ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m\theta + ie^{-ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \right] \end{aligned} \quad (6.201)$$

where ϵ_m is the Neumann factor which is equal to 1 for $m = 0$ and equal to 2 otherwise.

Since it can be shown that

$$\sum_{m=0}^M \epsilon_m \cos m\theta = \frac{\sin(M + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \quad (6.202)$$

The two infinite sums approach $2\pi\delta(\theta - m)$

a scattered wave

$$\psi = e^{ikx} + \frac{e^{ikr}}{\sqrt{r}} f(r\theta) \quad (6.203)$$

here

$$\frac{e^{ikr}}{\sqrt{r}} f(r, \theta) = -i \sum_{m=0}^{\infty} \epsilon_m e^{\frac{1}{2}m\pi i - i\delta_m} \sin \delta_m H_m(kr) \cos m\theta \quad , \quad (6.204)$$

$H_m(kr)$ are Hermite polynomials and

$$\tan \delta_m = \frac{-J_m(ka)}{N_m(ka)} \quad (6.205)$$

($N_m(ka)$ are Bessel functions of the second kind). For large values of r , the wave function can be written in a manner similar to (6.201), except that the outgoing wave is modified by the phase shifts δ_m .

$$\psi \simeq \frac{1}{\sqrt{2\pi ik}} i \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \frac{e^{-ikr}}{\sqrt{r}} + \frac{e^{ikr}}{\sqrt{r}} f(r, \theta) \quad . \quad (6.206)$$

here

$$f(r, \theta) \simeq \frac{1}{\sqrt{2\pi ik}} \sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta \quad (6.207)$$

In the limit that $ka \gg m$ the phase shifts can be written as

$$\delta_m \simeq ka - \frac{\pi}{2} \left(m + \frac{1}{2} \right) \quad . \quad (6.208)$$

In the limit of extremely large a (but $a < r$), the outgoing waves then behave as

$$f(r, \theta) \simeq \lim_{M \rightarrow \infty} -i \frac{1}{\sqrt{2\pi ik}} e^{-2ika} \frac{\sin \left(M + \frac{1}{2} \right) (\theta - \pi)}{\sin \frac{1}{2} (\theta - \pi)} \quad (6.209)$$

here once again we see that the angular distribution goes as the delta function $\delta(\theta - \pi)$. The disk scatters the plane wave directly back, and a sharp shadow is produced. We see therefore, that in the limit of $ka \gg 1$, our measurement of coincidence works.

The differential cross section can in general be written as

$$\begin{aligned} \sigma &= |f(\theta)|^2 \\ &= \left| \sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta \right|^2 \end{aligned} \quad (6.210)$$

For $ka \gg 1$ (but still finite), (6.210) can be computed using our expression for the phase shifts from (6.208)

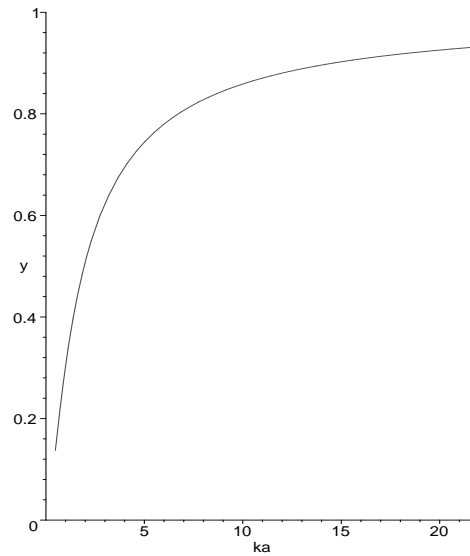


Figure 6.5: Phase shifts for coincidence detector ($\delta_1(k a) / \delta_0(k a)$ vs. $k a$)

depends on k . This is what we require then, for the probability of our measurement to succeed independently of the energy of the incoming particles. From a plot of δ_1 / δ_0 we see that this only occurs when $k a \gg 1$ (Figure 6.5). Our condition for an accurate measurement is therefore that $a \gg 1/k$. Since $\delta t_c \simeq a m / k_c \gg k$ fl θ $\frac{y}{m}$ θ m y m y θ $R a h$

Preparing a state ψ_c corresponds to preparing a single particle in two dimensions which always arrives inside a region $\delta r = p\delta t_c/m$ of the origin. In other words, suppose we were to set up a detector of size δr at the origin. If a state ψ_c exists, then it would always trigger the detector at some later time.

Our definition of coincidence requires that the state ψ_c

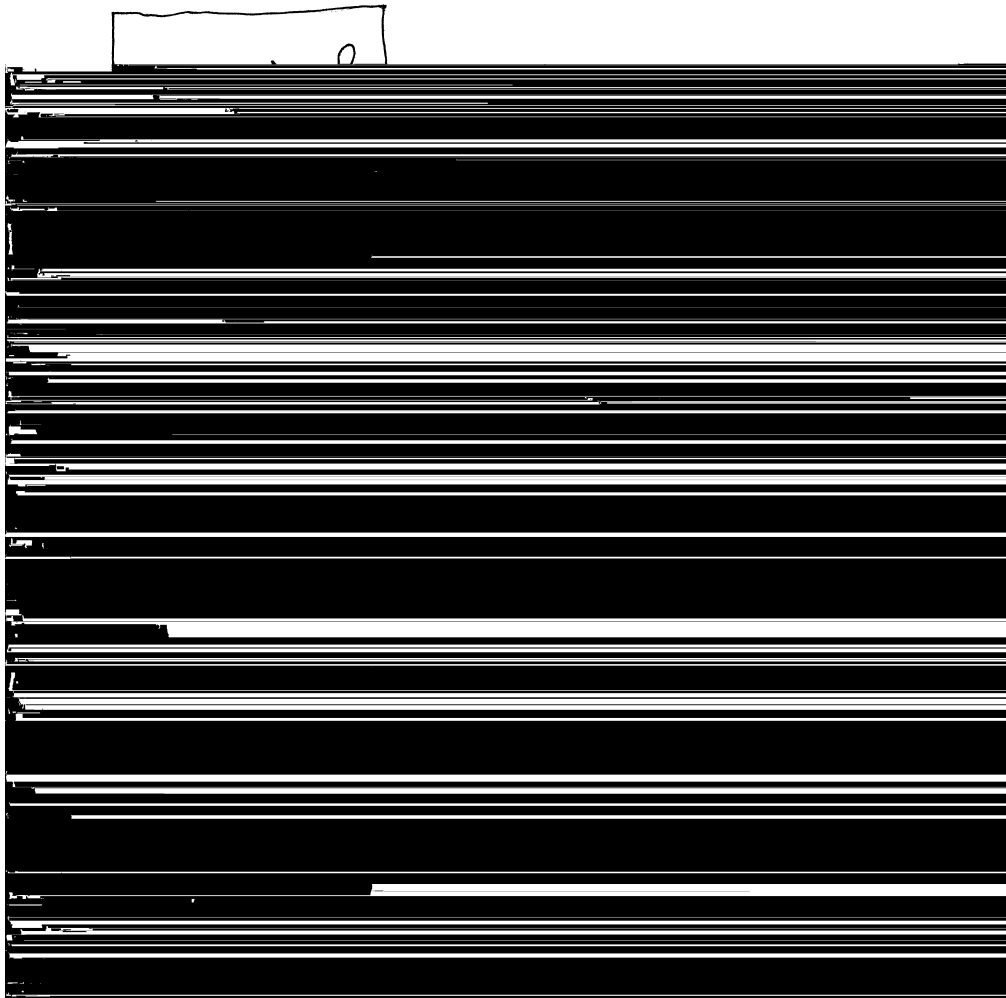
to do this to arbitrary accuracy, without affecting the system. In some sense, one may not be able to determine which way the light cone points.

One could use the arrival of arbitrarily energetic particles in order to denote space-time events, and although one can increase the energy of the particles in order to increase the accuracy with which one is able to measure the order of events, at some point the energy of the particles will effect the curvature of the neighboring space time.

Chapter 7

Conclusion

Schrödinger's cat



We have argued that time plays a unique role in quantum mechanics. It is unlike other observables and one cannot naively assume it to be measurable. We have examined a number of different types of measurements of the time of an event, including measurements which involve continual monitoring of the system, coupling to physical clocks, measuring of current operators, and time-of-arrival operators. These various types of measurements given different results, and there does not appear to be any canonical method for measuring the time of an event.

In the context of the time-of-arrival t_A , we have found a basic limitation on the accuracy (as opposed to uncertainty) that t_A can be determined reliably: $\delta t_A > 1/\bar{E}_k$. This limitation is quite different in origin from that due to the uncertainty principle; here it applies to the inference of the value of time for a *single* event. Furthermore, unlike the kinematic nature of the uncertainty principle, in our case the limitation is essentially dynamical in its origin; it arises when the time-of-arrival is measured by means of a continuous interaction between the measuring device and the particle.

While we know of no formal proof that this relation holds for time-of-arrival, our arguments are fairly general in nature. For the case of traversal time, we have argued that the limitation does not depend on any particular measurement procedure.

We have also argued that monitoring whether the particle is at the location of arrival x_A at various times, and also measuring the current operator, do not allow one to construct a probability distribution which one could interpret as representing the probability that the particle will arrive at a certain time.

We would also like to stress that continuous measurements differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. While the time-of-arrival operator is a formally constructed operator which can be measured by an impulsive von-Neumann interaction, continuous measurements are much closer to actual experiments. Furthermore, we have seen that the result of these two measurements do

not need to agree. In particular, at high accuracy, continuous measurements give rise to entirely different behavior – the particle never arrives. The time-of-arrival on the other hand, can be measured to any accuracy.

However, the time-of arrival operator is not self-adjoint. Attempts to modify the time-of-arrival operator in such a way as to make it self-adjoint result in the problem that the particle does not arrive on time with probability 1/2. Operators which classically might give the time of an event cannot be given a physical interpretation. While several authors [9][25] have maintained that the problems with defining an operator for the time of an event are technical, and can be circumvented by slightly modifying these operators. We have argued that probabilities in time are fundamentally different from traditional probabilities in quantum mechanics, and that there is a limitation on these measurements. As is the case with “time operators” [20] in closed quantum systems, the time-of-arrival operator has a somewhat limited physical meaning.

We have also seen that one cannot determine the temporal ordering of events to arbitrary accuracy. The limitation on these measurements is once again given by $1/\bar{E}$ here \bar{E} is the typical total energy of the system. Nor can one prepare a two particle system in a state in which the two particles always arrive within a time $1/\bar{E}$ of each other.

However as with most research, this thesis raises more questions than it answers. Does a formalism exist where time is an element of reality? If not, does there exist a proof of the minimum inaccuracy bounds we have proposed? More intriguing, are some of the connections between this research, and the problem of time in quantum gravity and quantum cosmology (effly discussed the canonical approach to quantizing gravity. One immediately encounters the problem that relative to the external parameter time in the Schrödinger equation, the state of the universe does not evolve. This is because the

system must satisfy constraints which are equivalent to reparametrization of the time variable.

The situation is somewhat analogous to being inside a box, and having some external observer weigh the box with high accuracy [40]. In order to keep the box at this fixed weight, the external experimenter cannot measure observables which evolve in time.

Quantum

These examples, which arise out of trying to understand quantum gravity, have led us to examine the role that time plays in ordinary quantum

Bibliography

- [1] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, W.G. Unruh, Phys. Rev. A, **57**, 4130, 1998 (quant-ph/9709031).

Phys. Rev. A.; J. Leon, J. Phys. A30 (1997) 4791 (quant-ph/9608013); V. Delgado and J. G. Muga, Phys. Rev. A **56**, 3425 (1997) (quant-ph/9704010). Ph. Blanchard, A. Jadczyk, quant-ph/9602010; Ph. Blanchard, A. Jadczyk, (quant-ph/9702019); V. Delgado, Phys. Rev. A **57** 762 (1988) (quant-ph/9709037); J.G. Muga, J.P. Palao, C.R. Leavens, (quant-ph/9803087) C.R. Leavens, "Time of arrival in quantum and Bohmian mechanics", Phys. Rev. A, **58**, 849, (1998); V. Olkhovsky, E. Recami, A. Gerasimchuk, Nuovo Cimento, **22**

- [22] See for example, C. Rovelli, Phys. Rev. D **42** 2638-2646 (1990); C. Rovelli, Phys. Rev. D **43**, 442 (1991)
- [23] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932) p. 195, [English translation: *Mathematical Foundations of Quantum Mechanics*, trans E.T. Beyer (Princeton University Press, Princeton, 1995) p.366]; B. Misra and E.C.G. Sudarshan, J. Math. Phys. **18**, 756 (1977),
- [24] H. Salecker and E. P. Wigner, Phys. Rev. **109**, 571 (1958).
- [25] J.G. Muga, S. Brouard, and D. Macias, Ann. Phys, **240**, 351 (1995)
- [26] M. D. Srinivas and R. Vijayalakshmi, Pramana, **16**, 173 (1981); Holevo, A.S. (1982). Probabilistic and statistical aspects of quantum theory, North-Holland Publ, Amsterdam; P. Busch, M. Grabowski, P. Lahti, Phys Let A **191** 357 (1994); R. Giannitrapani, (quant-ph/9611015)
- [27] Y. Aharonov, B. Reznik, quant-ph/9704001
- [28] C. Rovelli, Found. Phys. **28** (1998) 1031-1043, quant-ph/902020
- [29] Bracken A.J. and Melloy G.F., J. Phys. A: Math. Gen **27** (1994) 2197
- [30] G. Muga, C.R. Leavens, J. Palao, Phys. Rev. A **58** (1998)
- [31] J.M. Jauch, J.P. Marchand, Helv. Phys. Acta **40** (1967), 217; F.T. Smith, Phys. Rev. **118**, 349
- [32] W.G. Unruh hep-th/9410168
- [33] B. Mielnik, Found. Phys. **24**, 1113 (1994)
- [34] V. Delgado, J.G. Muga, Phys. Rev. A **56** 3425 (1997)
- [35] J. Palao, J. Muga, R. Sala, Phys. Rev. Lett **80** (1998) 5469-5472; (quant-ph/9805035)
- [36] N. Yamada, S. Takagi Prog. Theor. Phys, **85**, 985 (1991); **86**, 599 (1991); **87**, 77 (1992)
- [37] C.R. Leavens, "Time of arrival in quantum and Bohmian mechanics", Phys. Rev. A **58**, 849 (1998)
- [38] E. Arthurs and J.L. Kelly, Bell Syst. Tech. J. **44**, 1153 (1965)
- [39] N. Margolus, L. Levitin, Physica D **120** (1998) 188-195, quant-ph/9710043
- [40] G. Unruh, and G.I. Opat, Am. J. Phys. **47**, 743 (1979)

- [41] This is rather well known, but the suggestion of using this idea on an unsuspecting cat was suggested to me by Benni Reznik (private communication).
- [42] Y. Aharonov and B. Reznik, "Weighing" a Closed System and the Time-energy Uncertainty Principle, quant-ph/9906030
- [43] Y. Aharonov, and D. Bohm, Phys. Rev. **122**, 1649 (1961)
- [44] Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (San Diego: Academic Press, 1980). Aharonov and Bohm, Phys. Rev. **115**, 485 (1959).

Appendix A

Zero-Current Wavefunctions

One interesting aspect of the detector discussed in Section 3.3.2, is that while it can be used for wave-packets arriving from the left or the right, it will not always be triggered if the wavefunction is a coherent superposition of right and left moving modes. Consider for example, the superposition

$$\psi(x) = Ae^{ikx} + Ae^{-ikx}. \quad (\text{A.216})$$

One can easily verify that the current

$$j(x, t) = -i\frac{1}{2m} \left[\psi^*(x, t) \frac{\partial \psi(x, t)}{\partial x} - \frac{\partial \psi^*(x, t)}{\partial x} \psi(x, t) \right] \quad (\text{A.217})$$

is zero in this case. $|\psi(0, t)|^2$ is non-zero, although the state is not normalizable. As in eq. (3.63) this state evolves into

$$\langle x | \psi \rangle | \uparrow_z \rangle \rightarrow \frac{A}{\sqrt{2}} \left[(e^{ikx} + e^{-ikx}) | \uparrow_x \rangle + (e^{ikx} - e^{-ikx}) | \downarrow_x \rangle \right] \quad (\text{A.218})$$

Which, when rewritten in the σ_z basis, is just

$$A(e^{ikx} + e^{-ikx}) | \uparrow_z \rangle. \quad (\text{A.219})$$

i.e. the detector is never triggered.

This wavefunction is similar to the antisymmetric wavefunctions discussed by Yamada and Takagi in the context of decoherent histories [36] and Leavens [37] in the context of Bohmian mechanics, where also one finds that the particles never arrive. How to best treat these cases is an interesting open question.

Appendix B

Gaussian Wave Packet and Clocks

Using the simple model of Section 3.3.1 (3.44), we now calculate the probability distribution of a clock which measures the time-of-arrival of a Gaussian wave packet. We will perform the calculation in the limits when the clock is extremely accurate and extremely inaccurate. The wave function of the clock and particle is given by (3.52) and the distributions are both Gaussians given by (3.53). In the inaccurate limit, when $p_o \ll k$, $A_T \sim 1$. We trace over the position of the particle on the condition that the clock was triggered, ie. $x > 0$.

$$\begin{aligned} \rho(y, y)_{x>0} &= \int dx |\psi(x > 0, y, t)|^2 \\ &\simeq N^2 \int_{-\infty}^{\infty} dk dk' \int_0^{\infty} dp dp' dx g(k) g^*(k') f(p) f^*(p') e^{i(q-q')x + i(p-p')y - \frac{i(q^2 - q'^2)t}{2m}} \end{aligned} \quad (\text{B.220})$$

After a sufficiently long time, ie. $t \gg t_a$ the wave function has no support on the negative x-axis, and if $p_o > 1/\Delta y$, then it will not have support in negative p . We can thus integrate p and x over the entire axis. Integrating over x gives a delta-function in q . We can then integrate over p' to give

$$\rho(y, y)_{x>0} \simeq \frac{2\pi N^2}{m} \int dk dk' dp \sqrt{k^2 + 2mp} g(k) g^*(k') f(p) f^*(p + \frac{k^2 - k'^2}{2m}) e^{i(k'^2 - k^2) \frac{y}{2m}}$$

here we have used the fact that $\delta(f(z)) = \frac{\delta(z-z_o)}{f'(z=z_o)}$ when $f(z_o) = 0$. The square root term varies little in comparison with the exponential terms and can be replaced by its average value $\sqrt{k_o^2 + 2mp_o} \simeq k_o$. Integrating over p gives

$$\rho(y, y)_{x>0} \simeq \frac{2\pi N^2 k_o}{m} \sqrt{\frac{\pi}{2\Delta y^2}} \int dk dk' e^{-\frac{\Delta y^2}{8m^2} (k+k')^2 (k-k')^2} g(k) g^*(k') e^{i(k'^2 - k^2) \frac{y}{2m}}. \quad (\text{B.221})$$

Since $\Delta y k \gg 1$, for a wave packet peaked around k_o we can approximate the argument of the first exponential by $\frac{-\Delta y^2 k_o^2}{2m^2}(k - k')^2$. This allows us to integrate over k and k'

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \quad (\text{B.222})$$

here the width is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$.

As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial width Δy in clock position y . The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $1/dE = \frac{m}{kdk}$ and is given by $\frac{dx(y)m}{k_o}$ here $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the width in x arises because the wave packet is spreading as time increases, so that at later y , the wave packet is wider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles with larger mass.

When the clock is extremely accurate i.e. $p_o \gg k_o$ we have $A_{I'} \sim k\sqrt{\frac{2}{mp}}$.

$$\begin{aligned} \rho(y, y)_{x>0} &\simeq \frac{2N^2}{m} \int_{-\infty}^{\infty} dk dk' \int_0^{\infty} dp dp' dx \frac{kk'}{\sqrt{pp'}} g(k) g^*(k') f(p) f^*(p') e^{i(q-q')x + i(p-p')y - \frac{i(q^2 - q'^2)t}{2m}} \\ &\simeq \frac{4\pi N^2}{m} \int dk dk' dp \frac{kk'}{m} \sqrt{\frac{k^2 + 2mp}{p(p + k^2 - k'^2)}} \quad \text{m } \mathbf{R} \mathbf{M} \mathbf{I} \mathbf{I} \quad \mathbf{Q} \mathbf{M} \mathbf{I} \mathbf{I} \end{aligned}$$

here the width $\tilde{\gamma}(y) = \Delta x^2 + \left(\frac{y}{v}\right)^2$



Appendix C

Time-of-Arrival Eigenstates

We will now show that the eigenstates of [9]