Birch–Swinnerton-Dyer Study Group: The Parity Conjecture

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Motivation

Let E/K be an elliptic curve over a number field.

Theorem (Mordell–Weil)

$E(K) = \mathbb{Z}^{rank(E/K)} \times E(K)_{tors}$

What is rank(E/K)?

Conjecture (BSD 1)

- *$L(E/K, s)$ has analytic continuation to $\mathbb{C}$*
- *$rank(E/K) = ord_{s=1} L(E/K, s)$*

The Parity Conjecture

Conjecture (Functional equation)

$L(E/K, s) = \pm L(E/K, 2-s) \cdot (stu\square)$

The sign is called the global root number and is the product of local root numbers:

$$\omega(E/K) = \prod_{\nu} \omega_\nu(E/K)$$

$\omega(E/K) = +$

The Parity Conjecture

Conjecture (Functional equation)

$L(E/K, s) = \pm L(E/K, 2 - s) \cdot ($

The Local Root Number

The local root number is defined using the theory of "local" -factors.

Theorem (Langlands–Deligne)

There is a unique definition of local -factors satisfying the following:

- *Multiplicativity*
- *Inductivity in degree 0*
- *Quasi-characters*
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Example 1

Theorem

$$\omega_\nu(E/K) = \begin{cases} -1 & K_\nu = \mathbb{R} \text{ or } \mathbb{C} \\ +1 & E/K_\nu \text{ has good reduction} \\ -1 & E/K_\nu \text{ has split multiplicative reduction} \\ +1 & E/K_\nu \text{ has non-split multiplicative reduction} \end{cases}$$

Let $K = \mathbb{Q}$

Example 2

Now let $K = \mathbb{Q}(\sqrt{19})$ and $E : y^2 + xy = x^3 + x^2 - 95x - 399$, (114.a1)

- There are 2 infinite places
- There is 1 prime above 2, reduction is non-split multiplicative
- There are 2 primes above 3, reduction is non-split multiplicative
- There is 1 prime above 19, reduction is split multiplicative

$$\omega(E/K) = (-1)^2 \cdot +1 \cdot (+1)^2 \cdot -1 = -1$$

Have

$$\mathrm{rank}(E/K) = \mathrm{rank}(E/\mathbb{Q}) + \mathrm{rank}(E_{19}/\mathbb{Q})$$

where

$$E_{19} : y^2 = x^3 - x^2 - 551728x - 157527872 \text{ and } \mathrm{rank}(E_{19}/\mathbb{Q}) = 1$$